No Universal Multiplicative FDR Bound for the Benjamini–Hochberg Procedure with Correlated Two-Sided Gaussian Tests

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Abstract

We study the worst-case false discovery rate of the Benjamini–Hochberg procedure applied to two-sided Gaussian p-values when the correlation matrix is otherwise unrestricted. Dobriban [2026] shows that BH does not always control the FDR at its nominal level. An analogous folklore conjecture is that BH controls the FDR up to a universal multiplicative constant. We prove that this conjecture is false. In particular, we construct Gaussian models for which the inflation factor $\mathrm{FDR}(\mathrm{BH}_q)/q$ diverges as $q \downarrow 0$. More precisely, for every $q \in (0,1)$, the supremum over the number of hypotheses, mean vector, and correlation matrix is at least an explicit lower bound $\ell(q) > q$, and, as $q \downarrow 0$,

$$\ell(q) = \frac{q\sqrt{\log(1/q)}}{2\sqrt{\pi}} + c_\ell q + o(q), \qquad c_\ell = 0.6492828\ldots.$$

Finally, for a broad class of common-factor Gaussian models with arbitrary means and loadings, we prove the upper bound $\mathrm{FDR}(\mathrm{BH}_q) = O(q\sqrt{\log(1/q)})$. This upper bound has the same order as $\ell(q)$ when $q \downarrow 0$, so the rate $q\sqrt{\log(1/q)}$ is sharp for this class.

1 Introduction

We observe a Gaussian vector $T = (T_1, \ldots, T_m)$ with mean $\boldsymbol{\theta}$ and covariance matrix Σ, where $\Sigma_{ii} = 1$ for all i. The null hypotheses are $H_i : \theta_i = 0$, and their two-sided Gaussian p-values are $P_i = p(|T_i|)$, where

$$p(x) = 2\overline{\Phi}(x) = 2(1 - \Phi(x)).$$

Here Φ is the cumulative distribution function (CDF) of the standard normal distribution $N(0,1)$, and $\phi = \Phi'$ is its density. We use the convention $p^{-1}(0) = \infty$. The Benjamini–Hochberg procedure [Benjamini and Hochberg, 1995] at level q orders the p-values as

$$P_{(1)} \le \cdots \le P_{(m)}$$

and rejects the hypotheses corresponding to

$$P_{(1)}, \ldots, P_{(R)}, \quad R = \max\{1 \le k \le m : P_{(k)} \le qk/m\},$$

with $R = 0$ if the set is empty. If V is the number of rejected true nulls, then the false discovery proportion (FDP) and false discovery rate (FDR) are

$$\mathrm{FDP} = \frac{V}{R \vee 1}, \quad \mathrm{FDR} = \mathbb{E}(\mathrm{FDP}).$$

A long-standing conjecture in the FDR literature is that BH controls the FDR at its nominal level q, regardless of the correlation matrix [Reiner-Benaim, 2007, Benjamini, 2010, Roux, 2018, Sarkar, 2023, Sarkar and Zhang, 2025]. Benjamini [2010, p. 407] summarized the evidence as follows:

> "The modification to general dependence is often not needed: convincing simutheoretical evidence indicates that the same holds for two-sided z-tests with any correlation structure."

Benjamini then noted that a complete theoretical proof was still missing. These works stated or supported this conjecture. However, recently, Dobriban [2026] found a counterexample where $\mathrm{FDR} > 0.0104$ when $q = 0.01$.

In this paper, we ask a more refined question: how large can the FDR be as a function of q? Formally, we seek a lower bound $\ell(q)$ such that

$$\sup_{\substack{m\in\mathbb{Z}_+,\ \boldsymbol{\theta}\in\mathbb{R}^m,\ \Sigma\in\mathbb{S}_+^m \\ \Sigma_{ii}=1,\ 1\le i\le m}} \mathrm{FDR}(\mathrm{BH}_q) \ge \ell(q). \tag{1}$$

Here $\mathbb{Z}_+ = \{1, 2, \ldots\}$ and $\mathbb{S}_+^m$ denotes the cone of $m \times m$ symmetric positive semidefinite matrices. A relaxed folklore conjecture is that the worst-case FDR in (1) is bounded above by Cq for some universal constant $C > 0$. However, Proposition 2.2 shows that the lower bound constructed here satisfies, as $q \downarrow 0$,

$$\ell(q) = \frac{q\sqrt{\log(1/q)}}{2\sqrt{\pi}} + c_\ell q + o(q), \qquad c_\ell = 0.6492828\ldots.$$

In particular, the FDR inflation factor FDR/q can diverge as $q \downarrow 0$.

Moreover, we derive an exact formula for $\ell(q)$. We prove that $\ell(q) > q$ for every $q \in (0, 1)$. Instead of seeking a counterexample for a given nominal level q, our result disproves the conjecture that BH controls the FDR for correlated two-sided Gaussian tests. We plot the absolute deviation $\ell(q) - q$ and relative inflation $\ell(q)/q$ in Figure 1.

Unlike the familiar failures of FDR control for correlated one-sided Gaussian tests under negative dependence [Hochberg and Rom, 1995, Samuel-Cahn, 1996, Chi et al., 2025], our construction uses covariance matrices whose entries are all positive. For one-sided Gaussian p-values, nonnegative correlations imply PRDS, and BH therefore controls the FDR [Benjamini and Yekutieli, 2001]. The two-sided map $T_i \mapsto 2\overline{\Phi}(|T_i|)$, however, is not monotone and need not preserve PRDS. Indeed, because the construction here yields $\mathrm{FDR} > q$, its two-sided p-values cannot be PRDS on the true nulls. In particular, Benjamini and Yekutieli [2001] establish PRDS for positively correlated Gaussian test statistics in the one-sided case, whereas their corresponding PRDS guarantee for two-sided p-values is limited to independence.

The lower-bound rate $q\sqrt{\log(1/q)}$ should be compared with the FDR-linking theorem of Su [2018]. Positive regression dependence within nulls (PRDN) relaxes positive regression dependence on a subset (PRDS) by imposing the relevant positive dependence condition only on the null p-values. Under PRDN, the FDR-linking argument gives an $O(q\log(1/q))$ bound. Section 3 proves a sharper bound for the common-factor Gaussian class considered here:

$$\mathrm{FDR}(\mathrm{BH}_q) = O(q\sqrt{\log(1/q)}),$$

uniformly over the number of hypotheses, means, and loadings. Thus, in this common-factor Gaussian class, the upper-bound rate improves on the general PRDN linking bound and matches the asymptotic order of $\ell(q)$ as $q \downarrow 0$. Moreover, as explained in Remark 3.2, the two-sided null p-values in this class satisfy the PRDN property by the absolute-value Gaussian MTP_2 criterion of Karlin and Rinott [1981]. Hence the upper bound sharpens the general PRDN-based linking rate within this common-factor subclass.

2 Lower bound

2.1 Main result

$$M(y) = \sup_{a \geq 0} e^{-a^2/2} \cosh(ay). \tag{2}$$

Appendix A identifies the likelihood-ratio envelope: $M = 1$ on $[0, 1]$, and M is continuous, strictly increasing, and unbounded on $(1, \infty)$. Hence there is a unique $y_q > 1$ such that

$$qM(y_q) = 1. \tag{3}$$

Set

$$\ell(q) = q\Phi(1) + q \int_1^{y_q} M(y)\phi(y)\, \mathrm{d}y + \overline{\Phi}(y_q). \tag{4}$$

The next theorem shows that the lower bound can be approached arbitrarily closely by finite-dimensional Gaussian models with positive definite correlation matrices.

Theorem 2.1. *For $T \sim N(\theta, \Sigma)$, let $\mathrm{FDR}_{\theta,\Sigma}(\mathrm{BH}_q)$ denote the FDR of BH_q applied to T. For every $q \in (0, 1)$ and $\delta > 0$, there are $N < \infty$, $\theta \in \mathbb{R}^N$, and a positive definite correlation matrix Σ such that*

$$\mathrm{FDR}_{\theta,\Sigma}(\mathrm{BH}_q) > \ell(q) - \delta. \tag{5}$$

Consequently,

$$\sup_{\substack{N \in \mathbb{N},\ \theta \in \mathbb{R}^N, \\ \Sigma \succ 0,\ \mathrm{diag}(\Sigma) = \mathbf{1}_N}} \mathrm{FDR}_{\theta,\Sigma}(\mathrm{BH}_q) \geq \ell(q). \tag{6}$$

The next proposition gives the small-q expansion of the lower bound.

Proposition 2.2. *As $q \downarrow 0$,*

$$\ell(q) = \frac{q\sqrt{\log(1/q)}}{2\sqrt{\pi}} + c_\ell q + o(q),$$

where

$$\begin{aligned} c_\ell &= \Phi(1) - \frac{1}{2\sqrt{2\pi}} + \int_1^\infty \left\{ M(y)\phi(y) - \frac{1}{2\sqrt{2\pi}} \right\} \mathrm{d}y \\ &= 0.6492828\ldots. \end{aligned}$$

Proposition 2.3 further shows that $\ell(q) > q$ for every $q \in (0, 1)$. Consequently, BH fails to control the FDR at every nontrivial nominal level in the correlated two-sided Gaussian setting.

Proposition 2.3. *For every $q \in (0, 1)$,*

$$\ell(q) > q.$$

Figure 1 plots $\ell(q) - q$ and $\ell(q)/q$ for $q = 0.005k$, $k = 1, \ldots, 200$. Table 1 reports the lower bound $\ell(q)$ and relative inflation $\ell(q)/q$ at commonly used nominal levels.

q	0.01	0.05	0.10	0.20
$\ell(q)$	0.0136	0.0629	0.1209	0.2314
$\ell(q)/q$	1.3553	1.2571	1.2094	1.1570

Table 1: The lower bound $\ell(q)$ and relative inflation $\ell(q)/q$ at commonly used nominal levels.

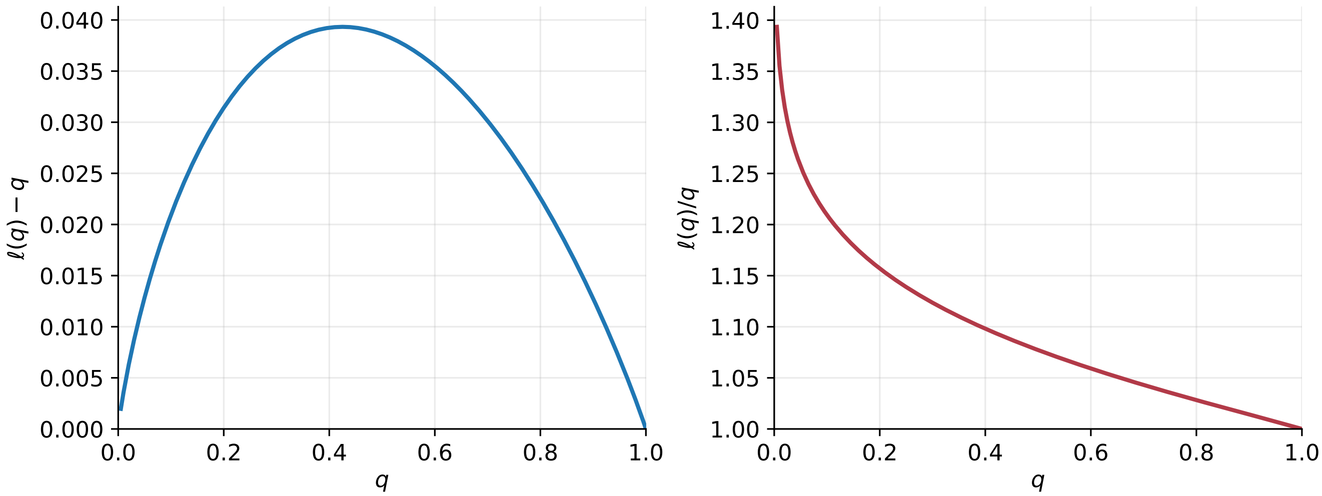

Figure 1: Numerical evaluation of $\ell(q) - q$ and $\ell(q)/q$ over the grid $q = 0.005k$, $k = 1, \dots, 200$.

2.2 Roadmap

Fix $q \in (0,1)$ and a design law ν_q. Let $\Omega_N = ((\Theta_1, B_1), \dots, (\Theta_N, B_N))$, where the pairs are i.i.d. draws from ν_q, independently of $Z, \varepsilon_1, \dots, \varepsilon_N \overset{\text{i.i.d.}}{\sim} N(0,1)$, and set

$$T_i = \Theta_i + B_i Z + \sqrt{1 - B_i^2}\, \varepsilon_i, \quad i = 1, \dots, N.$$

Conditional on a realization of Ω_N, this is a finite deterministic Gaussian design. Since BH is a step-up procedure, its rejection set is $\{i : |T_i| \geq \widehat{C}_N\}$. We will show in Subsection 2.4 that, conditional on $Z = z$, $\widehat{C}_N \to C(z)$ and $\mathrm{FDP}_N \to D(z)$. Consequently,

$$\mathbb{E}_{\Omega_N}[\mathrm{FDR}_{\Omega_N}(\mathrm{BH}_q)] = \mathbb{E}_{\Omega_N, Z, \varepsilon}[\mathrm{FDP}_N] \longrightarrow \mathbb{E}[D(Z)].$$

Thus, for any $\delta > 0$ and all sufficiently large N, at least one deterministic realization ω_N of Ω_N satisfies

$$\mathrm{FDR}_{\omega_N}(\mathrm{BH}_q) \geq \mathbb{E}[D(Z)] - \delta.$$

We next construct a family of laws $\nu_{q,r}$. Write D_r for the conditional FDP limit associated with $\nu_{q,r}$. For every $z \notin \{-1, -y_q\}$, we prove that $D_r(z) \to D_*(z)$ as $r \downarrow 0$, where

$$D_*(z) = \begin{cases} q, & z \geq -1, \\ qM(-z), & -y_q < z < -1, \\ 1, & z \leq -y_q. \end{cases}$$

Because $0 \leq D_r, D_* \leq 1$, dominated convergence then gives

$$\mathbb{E}[D_r(Z)] \longrightarrow \mathbb{E}[D_*(Z)] = \ell(q).$$

The argument therefore takes the iterated limit $\lim_{r \downarrow 0} \lim_{N \to \infty}$: first $N \to \infty$ for each fixed r, and then $r \downarrow 0$. To obtain a single finite example, we choose r sufficiently small and then N sufficiently large. A small independent Gaussian perturbation finally makes its covariance matrix positive definite.

2.3 Population quantities

Put $u_q = p^{-1}(q)$. Let ν_q be a probability measure on $\mathbb{R} \times [-1, 1]$. Write

$$\pi_0 = \nu_q(\{0\} \times [-1, 1]), \tag{7}$$

and assume $\pi_0 > 0$.

For $(\theta, b) \in \mathbb{R} \times [-1, 1]$, $z \in \mathbb{R}$, and $s \in (0, 1]$, let $u_s = p^{-1}(s)$ and define

$$F_{\theta,b,z}(s) = \mathbb{P}\{p(|T_i|) \le s \mid \Theta_i = \theta,\ B_i = b,\ Z = z\}.$$

If $|b| < 1$, then

$$F_{\theta,b,z}(s) = \overline{\Phi}\left(\frac{u_s - \theta - bz}{\sqrt{1 - b^2}}\right) + \overline{\Phi}\left(\frac{u_s + \theta + bz}{\sqrt{1 - b^2}}\right). \tag{8}$$

If $|b| = 1$, then

$$F_{\theta,b,z}(s) = \mathbf{1}\{p(|\theta + bz|) \le s\}. \tag{9}$$

Set $F_{\theta,b,z}(0) = 0$.

Conditional on $Z = z$, the CDFs of the mixture p-value and the null p-value are

$$R_z(s) = \int F_{\theta,b,z}(s)\, \nu_q(\mathrm{d}\theta, \mathrm{d}b), \tag{10}$$

$$R_{0,z}(s) = \frac{1}{\pi_0} \int_{\{0\} \times [-1,1]} F_{0,b,z}(s)\, \nu_q(\mathrm{d}\theta, \mathrm{d}b). \tag{11}$$

For $c \ge u_q$, define the normalized quantities

$$\bar{R}_z(c) = \frac{q R_z\{p(c)\}}{p(c)}, \tag{12}$$

$$\bar{R}_{0,z}(c) = \frac{q \pi_0 R_{0,z}\{p(c)\}}{p(c)}. \tag{13}$$

Then the limiting cutoff can be written as

$$C(z) = \inf\{c \ge u_q : \bar{R}_z(c) \ge 1\}. \tag{14}$$

Here and below, $\inf \varnothing = \infty$. The limiting conditional FDP is defined by

$$D(z) = \begin{cases} \bar{R}_{0,z}\{C(z)\}, & C(z) < \infty, \\ 0, & C(z) = \infty. \end{cases} \tag{15}$$

We call a finite cutoff a regular first crossing when it satisfies the following conditions.

Definition 2.4. A finite cutoff $C(z) > u_q$ is regular if

$$\sup_{u_q \le c \le C(z) - \rho} \bar{R}_z(c) < 1 \qquad \text{for every } 0 < \rho < C(z) - u_q, \tag{16}$$

$$\sup_{C(z) < c < C(z) + \rho} \bar{R}_z(c) > 1 \qquad \text{for every } \rho > 0, \tag{17}$$

and $c \mapsto \bar{R}_{0,z}(c)$ is continuous at $C(z)$.

2.4 Limiting cutoff and FDR

Let

$$\Omega_N = ((\Theta_1, B_1), \dots, (\Theta_N, B_N)), \quad (\Theta_i, B_i) \overset{\text{i.i.d.}}{\sim} \nu_q. \tag{18}$$

Since the distribution of $(T_1, \dots, T_N)$ is fully determined by Ω_N, we write $\mathrm{FDR}_{\theta,\Sigma}(\mathrm{BH}_q)$ as $\mathrm{FDR}_{\Omega_N}(\mathrm{BH}_q)$.

Define $P_i = p(|T_i|)$.

$$\widehat{R}_N(s) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}\{P_i \le s\}, \tag{19}$$

$$\widehat{R}_{0,N}(s) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}\{\Theta_i = 0,\ P_i \le s\}. \tag{20}$$

The following lemma gives a uniform conditional empirical-CDF bound.

Lemma 2.5. *For every $z \in \mathbb{R}$ and $t > 0$,*

$$\mathbb{P}\Big(\max\Big\{\|\widehat{R}_N - R_z\|_\infty, \|\widehat{R}_{0,N} - \pi_0 R_{0,z}\|_\infty\Big\} > t \,\Big|\, Z = z\Big) \le 4e^{-2Nt^2}. \tag{21}$$

Proof. Conditional on $Z = z$, the P_i are i.i.d. with CDF R_z. Also define

$$W_i = P_i \mathbf{1}\{\Theta_i = 0\} + 2\mathbf{1}\{\Theta_i \ne 0\}.$$

For $0 \le s \le 1$,

$$\mathbf{1}\{W_i \le s\} = \mathbf{1}\{\Theta_i = 0,\ P_i \le s\}, \quad \mathbb{P}(W_i \le s \mid Z = z) = \pi_0 R_{0,z}(s).$$

Apply the Dvoretzky–Kiefer–Wolfowitz inequality twice and take a union bound. □

The next lemma gives an exact characterization of the BH cutoff and FDP through the empirical CDFs.

Lemma 2.6. *Let R_N be the number of BH rejections and put*

$$\widehat{\tau}_N = \frac{qR_N}{N}. \tag{22}$$

If $R_N \ge 1$, then

$$\widehat{R}_N(\widehat{\tau}_N) = \frac{R_N}{N}, \quad \mathrm{FDP}_N = \frac{q\widehat{R}_{0,N}(\widehat{\tau}_N)}{\widehat{\tau}_N}. \tag{23}$$

With $\widehat{C}_N = p^{-1}(\widehat{\tau}_N)$,

$$\widehat{C}_N = \inf\left\{c \ge u_q : \frac{q\widehat{R}_N\{p(c)\}}{p(c)} \ge 1\right\}. \tag{24}$$

Then the BH rejection set is $\{i : |T_i| \ge \widehat{C}_N\}$. If $R_N = 0$, the set in (24) is empty and $\widehat{C}_N = \infty$.

Proof. Let $P_{(1)} \le \cdots \le P_{(N)}$. If more than R_N values were at most qR_N/N, then

$$P_{(R_N+1)} \le qR_N/N < q(R_N+1)/N,$$

contradicting the maximality of R_N. Hence $N\widehat{R}_N(\widehat{\tau}_N) = R_N$, and (23) follows. Since p is decreasing, the BH step-up definition is equivalent to (24). □

The next theorem establishes the limiting cutoff conditional on the common factor Z.

Theorem 2.7. *Assume that $C(z)$ is regular for almost every z. Then, for every z at which $C(z)$ is regular, conditional on $Z = z$,*

$$\widehat{C}_N \longrightarrow C(z), \quad \mathrm{FDP}_N \longrightarrow D(z) \tag{25}$$

in probability. Consequently,

$$\lim_{N\to\infty} \mathbb{E}_{\Omega_N}\left[\mathrm{FDR}_{\Omega_N}(\mathrm{BH}_q)\right] = \mathbb{E}[D(Z)], \tag{26}$$

where

$$\mathrm{FDR}_{\Omega_N}(\mathrm{BH}_q) = \mathbb{E}_{Z,\varepsilon}[\mathrm{FDP}_N \mid \Omega_N]\,. \tag{27}$$

Proof. Fix a regular z and $\rho, \tau > 0$. Since $D(z) = \bar{R}_{0,z}\{C(z)\}$ and $c \mapsto \bar{R}_{0,z}(c)$ is continuous at $C(z)$, choose

$$0 < \rho(z) < \min\{\rho, C(z) - u_q\}$$

so that

$$\sup_{C(z)-\rho(z)\leq c\leq C(z)+\rho(z)} |\bar{R}_{0,z}(c) - D(z)| \leq \tau. \tag{28}$$

Set $A(z) = C(z) - \rho(z)$. The left condition in Definition 2.4 gives $\sup_{u_q\leq c\leq A(z)} \bar{R}_z(c) < 1$, and its right condition permits a choice of $B(z) \in (C(z), C(z) + \rho(z))$ with $\bar{R}_z\{B(z)\} > 1$. Set

$$\gamma(z) = \frac{1}{2}\min\left\{1 - \sup_{u_q\leq c\leq A(z)} \bar{R}_z(c), \quad \bar{R}_z\{B(z)\} - 1\right\} > 0.$$

Put

$$E_N(z) = \max\left\{\|\widehat{R}_N - R_z\|_\infty, \|\widehat{R}_{0,N} - \pi_0 R_{0,z}\|_\infty\right\}.$$

Lemma 2.5 gives $E_N(z) \to 0$ in probability conditional on $Z = z$. For every $u_q \leq c \leq B(z)$,

$$\begin{aligned} \left|\frac{q\widehat{R}_N\{p(c)\}}{p(c)} - \bar{R}_z(c)\right| &\leq \frac{qE_N(z)}{p\{B(z)\}}, \\ \left|\frac{q\widehat{R}_{0,N}\{p(c)\}}{p(c)} - \bar{R}_{0,z}(c)\right| &\leq \frac{qE_N(z)}{p\{B(z)\}}. \end{aligned} \tag{29}$$

By the empirical curve we mean

$$c \longmapsto \frac{q\widehat{R}_N\{p(c)\}}{p(c)}.$$

On the event $qE_N(z)/p\{B(z)\} < \gamma(z)$, the definition of $\gamma(z)$ and (29) give

$$\begin{aligned} \sup_{u_q\leq c\leq A(z)} \frac{q\widehat{R}_N\{p(c)\}}{p(c)} &\leq \sup_{u_q\leq c\leq A(z)} \bar{R}_z(c) + \frac{qE_N(z)}{p\{B(z)\}} \\ &< 1 - 2\gamma(z) + \gamma(z) < 1, \end{aligned}$$

whereas

$$\frac{q\widehat{R}_N\{p(B(z))\}}{p(B(z))} \geq \bar{R}_z\{B(z)\} - \frac{qE_N(z)}{p\{B(z)\}} > 1 + 2\gamma(z) - \gamma(z) > 1.$$

Lemma 2.6 and (28) therefore give

$$A(z) < \widehat{C}_N \leq B(z), \qquad |\,\mathrm{FDP}_N - D(z)| \leq \tau + \frac{qE_N(z)}{p\{B(z)\}}.$$

Because $\rho, \tau > 0$ were arbitrary, this proves (25) at every regular z.

For completeness, the quantities in the expectation can be taken to be measurable without making a measurable selection of $B(z)$. The maps $(z,c) \mapsto \bar{R}_z(c)$ and $(z,c) \mapsto \bar{R}_{0,z}(c)$ are jointly Borel measurable, and the first is left-continuous in c. Define

$$C^\circ(z) = \inf\{c \in \mathbb{Q} : c \geq u_q,\ \bar{R}_z(c) > 1\}.$$

At every regular z, left continuity and (17) imply $C^\circ(z) = C(z)$. Thus C has a measurable version. Moreover, for $c < C(z)$, $\bar{R}_{0,z}(c) \leq \bar{R}_z(c) < 1$; continuity of the null curve at the crossing gives $0 \leq D(z) \leq 1$. Hence

$$D^\circ(z) = \begin{cases} \min[1, \bar{R}_{0,z}\{C^\circ(z)\}], & C^\circ(z) < \infty, \\ 0, & C^\circ(z) = \infty, \end{cases}$$

is a measurable $[0,1]$-valued version of $D(z)$ at almost every z. Use these versions below. Conditional convergence in probability, together with boundedness, now yields

$$\mathbb{E}[|\operatorname{FDP}_N - D(z)| \mid Z = z] \longrightarrow 0 \quad \text{for almost every } z.$$

Dominated convergence and the tower property give

$$\mathbb{E}_{\Omega_N, Z, \varepsilon}[|\operatorname{FDP}_N - D(Z)|] \longrightarrow 0,$$

and a second application of the tower property yields (26). □

Lastly, we show that Σ can be made positive definite by an independent Gaussian perturbation. For a fixed realization $\omega = ((\theta_i, b_i))_{i=1}^N$, let $\xi \sim N(0, I_N)$ be independent and define

$$T_i^{(\eta)} = \theta_i + \sqrt{1-\eta^2}\left(b_i Z + \sqrt{1-b_i^2}\,\varepsilon_i\right) + \eta \xi_i, \quad 0 < \eta < 1. \tag{30}$$

Then

$$\Sigma_{\omega,\eta} = (1-\eta^2)\left[bb^\top + \operatorname{diag}(1-b_1^2, \ldots, 1-b_N^2)\right] + \eta^2 I_N. \tag{31}$$

Lemma 2.8. *For every fixed ω,*

$$\operatorname{diag}(\Sigma_{\omega,\eta}) = \mathbf{1}_N, \quad \Sigma_{\omega,\eta} \succeq \eta^2 I_N, \tag{32}$$

and

$$\operatorname{FDR}_\omega^{(\eta)}(\mathrm{BH}_q) \longrightarrow \operatorname{FDR}_\omega(\mathrm{BH}_q) \quad (\eta \downarrow 0). \tag{33}$$

Proof. Equation (32) is immediate from (31). Under the coupling (30), $T^{(\eta)} \to T$ almost surely. Define $P_i^{(\eta)} = p(|T_i^{(\eta)}|)$. Then $P_i^{(\eta)} \to P_i = p(|T_i|)$ almost surely. Each P_i has a continuous marginal law, so

$$\mathbb{P}\left\{P_i \in \left\{\frac{qk}{N} : 1 \leq k \leq N\right\} \text{ for some } i\right\} = 0.$$

Off this event, the truth values of all finitely many comparisons $P_i^{(\eta)} \leq qk/N$ agree with those of $P_i \leq qk/N$ for all sufficiently small η. Hence the BH rejection count and rejected-null set are eventually constant. Bounded convergence proves (33). □

2.5 Construction of ν_q

We construct a family of laws $\nu_{q,r}$ indexed by $r \downarrow 0$. For $a, y \geq 0$, put

$$L_a(y) = e^{-a^2/2}\cosh(ay). \tag{34}$$

For $y > 1$, let

$$a(y) = \arg\max_{\alpha \geq 0} L_\alpha(y), \quad y(\alpha) = a^{-1}(\alpha), \quad \alpha > 0. \tag{35}$$

Set

$$a_q = a(y_q). \tag{36}$$

Appendix A establishes the properties of these functions that we use below.

The quantities $b_r, \tau_r, \mu_r, \pi_{0,r}$, and c_r are defined below. The law $\nu_{q,r}$ has the following three components:

type	mass	Θ	B
nulls	$\pi_{0,r}$	0	r
primary non-nulls	r	$c_r + y(b_r)$	1
secondary non-nulls	$\tau_r \mu_r(\,\mathrm{d}a)$	$a/r + y(a)$	$1, \quad a \in [b_r, a_q]$.

Equivalently, for every bounded measurable $f : \mathbb{R} \times [-1, 1] \to \mathbb{R}$,

$$\int f \,\mathrm{d}\nu_{q,r} = \pi_{0,r} f(0, r) + r f(c_r + y(b_r), 1) + \tau_r \int_{b_r}^{a_q} f\left(\frac{a}{r} + y(a), 1\right) \mu_r(\,\mathrm{d}a). \tag{37}$$

Definition of secondary non-nulls. For small r, set

$$b_r = r\, p^{-1}(r^2). \tag{38}$$

For the remaining definitions, take r small enough that $0 < b_r < a_q$. For $a \in [b_r, a_q]$, define

$$W_r(a) = \frac{[1 - qM\{y(a)\}]p(a/r)}{q}. \tag{39}$$

By Lemma A.1 and (133), the factor $1 - qM\{y(a)\}$ is nonnegative and decreasing on $[b_r, a_q]$; the factor $p(a/r)$ is decreasing in a. Hence W_r is continuous and decreasing, with $W_r(a_q) = 0$. Set

$$\tau_r := W_r(b_r) = \frac{[1 - qM\{y(b_r)\}]r^2}{q} = O(r^2). \tag{40}$$

We therefore call this the secondary-non-null block; its total mass is $O(r^2) = o(r)$. Let μ_r be the probability measure on $[b_r, a_q]$ determined by

$$\mu_r([a, a_q]) = \frac{W_r(a)}{\tau_r}, \quad b_r \leq a \leq a_q. \tag{41}$$

In particular, $\mu_r([b_r, a_q]) = W_r(b_r)/\tau_r = 1$. Equivalently,

$$\mu_r((a, b]) = \frac{W_r(a) - W_r(b)}{\tau_r}, \quad b_r \leq a < b \leq a_q. \tag{42}$$

Definition of nulls. Put

$$\pi_{0,r} = 1 - r - \tau_r.$$

Since $\pi_{0,r} + r + \tau_r = 1$, the masses in (37) define a probability measure.

Definition of primary non-nulls. For all sufficiently small r, $\pi_{0,r} > 0$. Define c_r by

$$p(c_r) = \frac{q(r + \tau_r)}{1 - q\pi_{0,r}}. \tag{43}$$

The numerator is positive. Since $\tau_r = O(r^2)$ and $\pi_{0,r} = 1 - r - \tau_r$,

$$1 - q\pi_{0,r} = 1 - q + qr + q\tau_r \longrightarrow 1 - q > 0, \quad q(r + \tau_r) \longrightarrow 0.$$

Thus the right side of (43) is positive and smaller than one for small r, so c_r is well-defined.

2.6 Population FDR analysis under $\nu_{q,r}$

2.6.1 Preliminaries

Apply Subsection 2.3 with $\nu_q = \nu_{q,r}$. Denote the resulting quantities by

$$R_{r,z}, \quad R_{0,r,z}, \quad \bar{R}_{r,z}, \quad \bar{R}_{0,r,z}, \quad C_r(z).$$

For $c \geq u_q$,

$$\begin{aligned}\bar{R}_{r,z}(c) = q\pi_{0,r}&\frac{R_{0,r,z}\{p(c)\}}{p(c)} \\ &+ \frac{qr}{p(c)}\mathbf{1}\{|c_r + y(b_r) + z| \geq c\} \\ &+ \frac{q\tau_r}{p(c)}\int_{b_r}^{a_q}\mathbf{1}\left\{\left|\frac{a}{r} + y(a) + z\right| \geq c\right\}\mu_r(\,\mathrm{d}a),\end{aligned} \tag{44}$$

and

$$\bar{R}_{0,r,z}(c) = q\pi_{0,r}\frac{R_{0,r,z}\{p(c)\}}{p(c)}. \tag{45}$$

Whenever $C_r(z)$ is regular, set

$$D_r(z) = \bar{R}_{0,r,z}\{C_r(z)\}. \tag{46}$$

Set $D_r(z) = 0$ otherwise. At every regular crossing, $0 \leq D_r(z) \leq 1$.

We state two lemmas that will be applied in the following analysis. The first lemma gives the order of b_r and c_r as $r \downarrow 0$.

Lemma 2.9. *As $r \downarrow 0$,*

$$b_r \downarrow 0, \quad \frac{b_r}{r} = p^{-1}(r^2) \to \infty, \quad p(b_r/r) = r^2, \quad y(b_r) \downarrow 1. \tag{47}$$

Moreover,

$$p(c_r) \sim \frac{qr}{1-q}, \quad c_r \to \infty, \quad rc_r \to 0, \quad \frac{b_r}{r} - c_r \to \infty. \tag{48}$$

Proof. The identity $p(b_r/r) = r^2$ follows directly from $b_r = r\,p^{-1}(r^2)$. Applying (142) in Lemma A.2 gives

$$\frac{b_r}{r} = p^{-1}(r^2) \sim \sqrt{4\log(1/r)} \to \infty, \quad b_r \sim 2r\sqrt{\log(1/r)} \to 0.$$

The last limit in (47) follows from Lemma A.1: $a(y)$ is continuous and strictly increasing from $(1,\infty)$ onto $(0,\infty)$, with $\lim_{y\downarrow 1} a(y) = 0$. Since $y(\cdot)$ is its inverse and $b_r \downarrow 0$, we have $y(b_r) \downarrow 1$.

By (43) and (40),

$$p(c_r) = \frac{q(r+\tau_r)}{1-q\pi_{0,r}} \sim \frac{qr}{1-q}.$$

Thus

$$\log\frac{1}{p(c_r)} = \log\frac{1}{r} + O(1).$$

Applying (142) in Lemma A.2 gives

$$c_r = p^{-1}\{p(c_r)\} \sim \sqrt{2\log(1/r)}.$$

Hence $c_r \to \infty$ and $rc_r \to 0$. Combining this with

$$\frac{b_r}{r} = p^{-1}(r^2) \sim \sqrt{4\log(1/r)}$$

gives

$$\frac{b_r}{r} - c_r \sim \left(2 - \sqrt{2}\right) \sqrt{\log(1/r)} \longrightarrow \infty.$$

□

The next lemma gives a uniform approximation to the normalized conditional FDP contribution from nulls.

Lemma 2.10. *For every $A > 0$ and $T \geq 0$, as $r \downarrow 0$,*

$$\sup_{|z|\leq T} \sup_{0\leq c\leq A/r} \left| \frac{R_{0,r,z}\{p(c)\}}{p(c)} - L_{rc}(|z|) \right| \longrightarrow 0. \tag{49}$$

Proof. For $0 < r < 1$, the null pair $(\Theta, B) = (0, r)$ gives

$$R_{0,r,z}\{p(c)\} = \mathbb{P}\{|rz + \sqrt{1-r^2}\,\varepsilon| \geq c\}.$$

Write

$$x_\pm = \frac{c \mp rz}{\sqrt{1-r^2}}.$$

Then

$$R_{0,r,z}\{p(c)\} = \overline{\Phi}(x_+) + \overline{\Phi}(x_-).$$

Fix $C > 0$. Uniformly for $|z| \leq T$ and $C \leq c \leq A/r$,

$$\begin{aligned} x_\pm - c &= c\left\{\frac{1}{\sqrt{1-r^2}} - 1\right\} \mp \frac{rz}{\sqrt{1-r^2}} \\ &= \frac{cr^2}{\sqrt{1-r^2}\{1+\sqrt{1-r^2}\}} \mp \frac{rz}{\sqrt{1-r^2}}. \end{aligned}$$

Therefore,

$$|x_\pm - c| \leq \frac{Ar}{\sqrt{1-r^2}\{1+\sqrt{1-r^2}\}} + \frac{Tr}{\sqrt{1-r^2}} = O_{A,T}(r).$$

Consequently, for all sufficiently small r, the shifts $x_\pm - c$ are uniformly bounded and $x_\pm \geq C/2$. The uniform expansion (141) in Lemma A.2 therefore gives remainders $\rho_{r,C,\pm}(c,z)$ such that

$$\begin{aligned} \frac{\overline{\Phi}(x_\pm)}{\overline{\Phi}(c)} &= \frac{c}{x_\pm} \exp\left\{-\frac{x_\pm^2 - c^2}{2}\right\} \{1 + \rho_{r,C,\pm}(c,z)\}, \\ -\frac{x_\pm^2 - c^2}{2} &= -\frac{(rc)^2}{2} \pm rcz + O_{A,T}(r^2). \end{aligned} \tag{50}$$

and

$$\lim_{C\to\infty} \limsup_{r\downarrow 0} \sup_{|z|\leq T} \sup_{C\leq c\leq A/r} \max_{\pm} |\rho_{r,C,\pm}(c,z)| = 0. \tag{51}$$

For each fixed $C > 0$, $c/x_\pm \to 1$ uniformly for $|z| \leq T$ and $C \leq c \leq A/r$ as $r \downarrow 0$. To identify the leading term, put $t = rc$, so that $0 \leq t \leq A$. Then

$$\frac{1}{2}\left[\exp\left\{-\frac{t^2}{2} + tz\right\} + \exp\left\{-\frac{t^2}{2} - tz\right\}\right] = e^{-t^2/2}\cosh(tz) = L_t(|z|).$$

Because t and z range over compact sets, these exponential factors are uniformly bounded. Thus, after summing the two tails in (50) and using $p(c) = 2\overline{\Phi}(c)$, the errors from $c/x_\pm - 1$ and from the $O_{A,T}(r^2)$

term vanish uniformly for each fixed C. Consequently, there is a constant $K_{A,T} < \infty$, independent of C, such that, for

$$\eta(C) = \limsup_{r\downarrow 0} \sup_{|z|\le T} \sup_{C\le c\le A/r} \left| \frac{R_{0,r,z}\{p(c)\}}{p(c)} - L_{rc}(|z|) \right|,$$

we have

$$\eta(C) \le K_{A,T} \limsup_{r\downarrow 0} \sup_{|z|\le T} \sup_{C\le c\le A/r} \max_{\pm} |\rho_{r,C,\pm}(c,z)|.$$

The right side tends to zero as $C \to \infty$ by (51). Hence $\eta(C) \to 0$.

It remains to control $0 \le c \le C$. On this range, $p(c) \ge p(C) > 0$. Also, uniformly over $0 \le c \le C$ and $|z| \le T$,

$$R_{0,r,z}\{p(c)\} = \overline{\Phi}\left(\frac{c - rz}{\sqrt{1-r^2}}\right) + \overline{\Phi}\left(\frac{c + rz}{\sqrt{1-r^2}}\right) = p(c) + O_{C,T}(r),$$

because the two arguments differ from c by $O_{C,T}(r)$ and $\overline{\Phi}$ has bounded derivative. Hence

$$\sup_{|z|\le T} \sup_{0\le c\le C} \left| \frac{R_{0,r,z}\{p(c)\}}{p(c)} - 1 \right| \longrightarrow 0.$$

Moreover, $L_{rc}(|z|) = e^{-(rc)^2/2}\cosh(rc|z|) \to 1$ uniformly for $|z| \le T$ and $0 \le c \le C$. Thus

$$\limsup_{r\downarrow 0} \sup_{|z|\le T} \sup_{0\le c\le A/r} \left| \frac{R_{0,r,z}\{p(c)\}}{p(c)} - L_{rc}(|z|) \right| \le \eta(C).$$

Letting $C \to \infty$ proves the claim. □

2.6.2 The regime $z > -1$

The next lemma locates the crossing created by the primary non-null.

Lemma 2.11. *For every compact $K \subset (-1,\infty)$ and every $H > 0$, uniformly for $z \in K$ and $|h| \le H$,*

$$\bar{R}_{r,z}\left(c_r + \frac{h}{c_r}\right) \longrightarrow q + (1-q)e^{h}. \tag{52}$$

Moreover, for every $h_0 > 0$,

$$\limsup_{r\downarrow 0} \sup_{z\in K} \sup_{u_q\le c\le c_r - h_0/c_r} \bar{R}_{r,z}(c) \le q + (1-q)e^{-h_0} < 1. \tag{53}$$

Consequently,

$$C_r(z) = c_r + o(c_r^{-1}), \quad D_r(z) \longrightarrow q, \tag{54}$$

uniformly on K. Moreover, for sufficiently small r, $C_r(z)$ is regular for every $z \in K$.

Proof. **Proof of (52).** Fix $H > 0$, take $z \in K$, and take $|h| \le H$. The primary non-null has $B = 1$ and $\Theta = c_r + y(b_r)$, so conditional on $Z = z$ its value is $c_r + y(b_r) + z$. Since $K \subset (-1,\infty)$ is compact,

$$\eta_K := \inf_{z\in K}(1+z) > 0.$$

By Lemma 2.9, $c_r \to \infty$ and $y(b_r) \downarrow 1$. For the fixed H, this implies $H/c_r \to 0$. Combining these limits, for sufficiently small r,

$$|y(b_r) - 1| \le \frac{\eta_K}{4}, \quad \frac{H}{c_r} \le \frac{\eta_K}{4}. \tag{55}$$

For $z \in K$ and $|h| \leq H$, the definition of η_K gives $z \geq -1 + \eta_K$, while (55) gives $y(b_r) \geq 1 - \eta_K/4$ and $-h/c_r \geq -H/c_r \geq -\eta_K/4$. Therefore

$$y(b_r) + z - \frac{h}{c_r} \geq \left(1 - \frac{\eta_K}{4}\right) + (-1 + \eta_K) - \frac{\eta_K}{4} = \frac{\eta_K}{2}, \quad z \in K, \ |h| \leq H.$$

Equivalently, $y(b_r) + z \geq h/c_r + \eta_K/2$. Hence the conditional value of the primary non-null satisfies

$$\begin{aligned} |c_r + y(b_r) + z| &= c_r + y(b_r) + z \\ &= c_r + \frac{h}{c_r} + \left\{y(b_r) + z - \frac{h}{c_r}\right\} \\ &\geq c_r + \frac{h}{c_r} + \frac{\eta_K}{2} > c_r + \frac{h}{c_r}, \end{aligned} \tag{56}$$

so the primary non-null is rejected at cutoff $c_r + h/c_r$, uniformly for $z \in K$ and $|h| \leq H$.

Next consider a secondary non-null indexed by $a \in [b_r, a_q]$. Its conditional value at $Z = z$ is $a/r + y(a) + z$. Since $a \geq b_r$, we have $a/r \geq b_r/r$. Also, Lemma A.1 shows that $y(\cdot)$, the inverse of the strictly increasing map $a(\cdot)$, is increasing; hence $y(a) \geq y(b_r)$. Together with $z \geq \inf_{u \in K} u$ and $h \leq H$, this gives

$$\frac{a}{r} + y(a) + z - \left(c_r + \frac{h}{c_r}\right) \geq \left(\frac{b_r}{r} - c_r\right) + y(b_r) + \inf_{u \in K} u - \frac{H}{c_r}. \tag{57}$$

Here $\frac{b_r}{r} - c_r \to \infty$ by (48), while

$$y(b_r) + \inf_{u \in K} u - \frac{H}{c_r} \longrightarrow 1 + \inf_{u \in K} u \tag{58}$$

because $y(b_r) \downarrow 1$ and $H/c_r \to 0$. Equations (48) and (58) show that the right side of (57) tends to $+\infty$. Hence it is positive for sufficiently small r, uniformly in $z \in K$, $|h| \leq H$, and $a \in [b_r, a_q]$. Thus every secondary non-null is rejected at cutoff $c_r + h/c_r$.

Lastly, we analyze the nulls. Since $|h| \leq H$ and $rc_r \to 0$ by (48),

$$r\left(c_r + \frac{h}{c_r}\right) = rc_r + \frac{rh}{c_r} \longrightarrow 0. \tag{59}$$

Let $T_K = \sup_{z \in K} |z| < \infty$. Since $c_r \to \infty$ by (48) in Lemma 2.9, while (59) gives a vanishing rescaled cutoff, we have

$$u_q \leq c_r + \frac{h}{c_r} \leq \frac{1}{r}$$

for all sufficiently small r, uniformly for $z \in K$ and $|h| \leq H$. Applying (49) in Lemma 2.10 with $A = 1$ and $T = T_K$ gives

$$\frac{R_{0,r,z}\{p(c_r + h/c_r)\}}{p(c_r + h/c_r)} = L_{r(c_r + h/c_r)}(|z|) + o(1).$$

Moreover,

$$L_{r(c_r+h/c_r)}(|z|) = \exp\left\{-\frac{r^2(c_r + h/c_r)^2}{2}\right\} \cosh(r(c_r + h/c_r)|z|) = 1 + o(1),$$

uniformly for $z \in K$ and $|h| \leq H$. Since $\pi_{0,r} = 1 - r - \tau_r \to 1$, it follows that

$$q\pi_{0,r} \frac{R_{0,r,z}\{p(c_r + h/c_r)\}}{p(c_r + h/c_r)} = q + o(1).$$

By (43),

$$\frac{q(r + \tau_r)}{p(c_r + h/c_r)} = (1 - q\pi_{0,r}) \frac{p(c_r)}{p(c_r + h/c_r)} \longrightarrow (1 - q)e^h$$

by (146) in Lemma A.2. This proves (52).

Proof of (53)**.** Fix $h_0 > 0$. Equation (56), evaluated at $h = h_0$, gives

$$|c_r + y(b_r) + z| > c_r + \frac{h_0}{c_r},$$

so the primary non-null is rejected at cutoff $c_r + h_0/c_r$. Equations (57), (48), and (58) give

$$\frac{a}{r} + y(a) + z > c_r + \frac{h_0}{c_r}, \quad a \in [b_r, a_q],$$

for sufficiently small r, uniformly in $z \in K$. Thus every secondary non-null is also rejected at cutoff $c_r + h_0/c_r$. Since rejection at cutoff c means $|T_i| \geq c$, all non-nulls are rejected for every $u_q \leq c \leq c_r + h_0/c_r$. Therefore, on this interval,

$$\bar{R}_{r,z}(c) = q\pi_{0,r}\frac{R_{0,r,z}\{p(c)\}}{p(c)} + \frac{q(r+\tau_r)}{p(c)}.$$

For $u_q \leq c \leq c_r - h_0/c_r$, Lemma 2.10 and $rc_r \to 0$ from (48) give

$$\sup_{z\in K}\sup_{u_q\leq c\leq c_r-h_0/c_r} q\pi_{0,r}\frac{R_{0,r,z}\{p(c)\}}{p(c)} \leq q + o(1).$$

Also $p(c) \geq p(c_r - h_0/c_r)$, so (43) and (146) in Lemma A.2 give

$$\sup_{u_q\leq c\leq c_r-h_0/c_r} \frac{q(r+\tau_r)}{p(c)} \leq \frac{q(r+\tau_r)}{p(c_r - h_0/c_r)} = (1 - q\pi_{0,r})\frac{p(c_r)}{p(c_r - h_0/c_r)} \longrightarrow (1-q)e^{-h_0}.$$

Taking the supremum over $z \in K$ and then the limsup in r proves (53).

Proof of (54) **and regularity of C_r.** By (53), for sufficiently small r,

$$\sup_{z\in K}\sup_{u_q\leq c\leq c_r-h_0/c_r} \bar{R}_{r,z}(c) < 1.$$

Thus no first crossing can occur at or before $c_r - h_0/c_r$, uniformly for $z \in K$. At the upper point, (52) with $h = h_0$ gives

$$\bar{R}_{r,z}\left(c_r + \frac{h_0}{c_r}\right) = q + (1-q)e^{h_0} + o(1) > 1.$$

Thus, for sufficiently small r, the first crossing satisfies

$$c_r - \frac{h_0}{c_r} < C_r(z) \leq c_r + \frac{h_0}{c_r}, \quad z \in K. \tag{60}$$

Since $h_0 > 0$ is arbitrary, this yields $C_r(z) = c_r + o(c_r^{-1})$, uniformly on K.

We next verify regularity as stated in Definition 2.4, with r fixed. Fix $H > 0$ and write

$$G_{r,z}(c) = \frac{R_{0,r,z}\{p(c)\}}{p(c)}.$$

On $|c - c_r| \leq H/c_r$, all non-nulls are rejected, uniformly over $z \in K$, and hence

$$\bar{R}_{r,z}(c) = q\pi_{0,r}G_{r,z}(c) + \frac{q(r+\tau_r)}{p(c)}. \tag{61}$$

Put $x_\pm = (c \mp rz)/\sqrt{1-r^2}$. Throughout this paragraph, the window is the local range $|c - c_r| \leq H/c_r$, with $z \in K$. Direct differentiation gives

$$G'_{r,z}(c) = \frac{\phi(c)}{\bar{\Phi}(c)}G_{r,z}(c) - \frac{\phi(x_+) + \phi(x_-)}{2\sqrt{1-r^2}\,\bar{\Phi}(c)}.$$

Uniformly on this window, Lemmas 2.10 and 2.9 give $G_{r,z}(c) = 1 + o(1)$. We also have $c \sim c_r \to \infty$. Since $rc_r \to 0$ and $|c - c_r| \le H/c_r$,

$$x_\pm^2 - c^2 = O_K(rc_r) + O(r^2 c_r^2) = o(1), \qquad \sqrt{1 - r^2} = 1 + o(1),$$

uniformly over the same window. Hence

$$\frac{\phi(x_\pm)}{\phi(c)} = \exp\left\{-\frac{x_\pm^2 - c^2}{2}\right\} = 1 + o(1),$$

and therefore

$$\frac{\phi(x_+) + \phi(x_-)}{2\sqrt{1 - r^2}\,\phi(c)} = \frac{1 + o(1) + 1 + o(1)}{2\{1 + o(1)\}} = 1 + o(1).$$

Finally, (137) in Lemma A.2 gives $\phi(c)/\overline{\Phi}(c) = c\{1 + o(1)\}$, uniformly for $|c - c_r| \le H/c_r$, and therefore

$$\frac{\phi(c)}{\overline{\Phi}(c)} \sim c_r.$$

Consequently,

$$\sup_{z \in K} \sup_{|c - c_r| \le H/c_r} |G'_{r,z}(c)| = o(c_r).$$

On the same window, write $c = c_r + h/c_r$, where $|h| \le H$. By (43),

$$\frac{q(r + \tau_r)}{p(c)} = (1 - q\pi_{0,r})\frac{p(c_r)}{p(c)},$$

and (146) in Lemma A.2 gives

$$\frac{p(c_r)}{p(c)} = \frac{p(c_r)}{p(c_r + h/c_r)} \longrightarrow e^h,$$

uniformly for $|h| \le H$. Since $e^h \ge e^{-H}$ on this range and $1 - q\pi_{0,r} \to 1 - q > 0$, for all sufficiently small r,

$$\inf_{|c - c_r| \le H/c_r} \frac{q(r + \tau_r)}{p(c)} \ge \frac{1 - q}{2} \cdot \frac{e^{-H}}{2}.$$

Thus we may take $m_H = (1 - q)e^{-H}/4 > 0$. Since

$$\frac{\mathrm{d}}{\mathrm{d}c}\frac{q(r + \tau_r)}{p(c)} = \frac{q(r + \tau_r)}{p(c)}\frac{\phi(c)}{\overline{\Phi}(c)},$$

Together with $\phi(c)/\overline{\Phi}(c) \sim c_r$, uniformly on the same window, this gives

$$\inf_{|c - c_r| \le H/c_r} \frac{\mathrm{d}}{\mathrm{d}c}\frac{q(r + \tau_r)}{p(c)} \ge \frac{m_H}{2} c_r$$

for all sufficiently small r. By contrast, $q\pi_{0,r}G'_{r,z}(c) = o(c_r)$ uniformly over $z \in K$ and $|c - c_r| \le H/c_r$. Differentiating (61) therefore shows that

$$\inf_{z \in K} \inf_{|c - c_r| \le H/c_r} \bar{R}'_{r,z}(c) > 0 \tag{62}$$

for all sufficiently small r.

Take, for example, $H = 1$. The cutoff bracket (60), applied with $h_0 = 1/2$, places $C_r(z)$ in the interior of this window, uniformly on K. The population rejection curve

$$c \longmapsto \bar{R}_{r,z}(c),$$

defined in (44), is continuous through $c = C_r(z)$ and is strictly increasing there by (62). Hence $\bar{R}_{r,z}\{C_r(z)\} = 1$, and $\bar{R}_{r,z}(c) > 1$ for points $c > C_r(z)$ arbitrarily close to $C_r(z)$. This proves (17) for every $\rho > 0$ at each fixed sufficiently small r. Moreover, all non-nulls are rejected on $[u_q, C_r(z)]$, so (44) has the same continuous form as (61) on this interval. By the definition of $C_r(z)$, $\bar{R}_{r,z}(c) < 1$ at every $c < C_r(z)$; compactness of $[u_q, C_r(z) - \rho]$ proves (16) for every $0 < \rho < C_r(z) - u_q$. Finally, the null contribution

$$c \longmapsto \bar{R}_{0,r,z}(c) = q\pi_{0,r}\frac{R_{0,r,z}\{p(c)\}}{p(c)},$$

displayed in (45), is smooth at $c = C_r(z)$. Thus $C_r(z)$ is regular, uniformly for $z \in K$, for all sufficiently small r.

Finally,

$$rC_r(z) = rc_r + o\left(\frac{r}{c_r}\right) \longrightarrow 0$$

by (48), uniformly on K. Consequently, for all sufficiently small r, $u_q \leq C_r(z) \leq 1/r$ uniformly on K, so (49) in Lemma 2.10, with $A = 1$, applies at $c = C_r(z)$. Its leading factor satisfies

$$L_{rC_r(z)}(|z|) = \exp\left\{-\frac{r^2C_r(z)^2}{2}\right\}\cosh\{rC_r(z)|z|\} \longrightarrow 1$$

uniformly on K. Hence

$$D_r(z) = q\pi_{0,r}\frac{R_{0,r,z}\{p(C_r(z))\}}{p(C_r(z))} = q + o(1),$$

uniformly on K. □

2.6.3 The regime $z \in (-y_q, -1)$

Let $y = -z$. For $1 < y < y_q$, put

$$\kappa(y) = a(y)y'\{a(y)\} > 0. \tag{63}$$

The positivity follows from Lemma A.1, since $a(y) > 0$ and $y'\{a(y)\} > 0$ for $1 < y < y_q$.

We first prove a lemma that is useful for the key result in this regime.

Lemma 2.12. *For every compact $I \subset (1, y_q)$ and every $H > 0$, there is an interval U_I whose closure is contained in $(0, a_q)$ and contains $\{a(y) : y \in I\}$. Define*

$$g_r(\tilde{a}) = \tilde{a} + ry(\tilde{a}). \tag{64}$$

For all sufficiently small r, the function g_r is strictly increasing on $[b_r, a_q]$, and in particular on U_I.

For all sufficiently small r, the equation

$$g_r\{\tilde{\alpha}_r(y, h)\} = g_r\{a(y)\} + rh \tag{65}$$

has a unique solution $\tilde{\alpha}_r(y, h) \in U_I$, uniformly for $y \in I$ and $|h| \leq H$, and

$$\tilde{\alpha}_r(y, h) = a(y) + O(r). \tag{66}$$

Conditional on $Z = -y$, a secondary non-null with $a = \tilde{a}$ is rejected at cutoff $a(y)/r + h$ if and only if

$$\tilde{a} \in [\tilde{\alpha}_r(y, h), a_q].$$

Finally,

$$\log\frac{p\{\tilde{\alpha}_r(y, h)/r\}}{p(a(y)/r + h)} = \kappa(y)h + o(1), \tag{67}$$

uniformly for $y \in I$ and $|h| \leq H$.

Proof. Fix a compact $I \subset (1, y_q)$ and $H > 0$ as in the statement. Let

$$A_I = \{a(y) : y \in I\}.$$

Because $I \subset (1, y_q)$ is compact, A_I is a compact subset of $(0, a_q)$. Choose an interval U_I whose closure is contained in $(0, a_q)$ and such that $A_I \subset U_I$. By Lemma A.1, $y(\cdot)$ is continuously differentiable and $y' > 0$ on $(0, a_q)$. Hence

$$g_r'(\tilde{a}) = 1 + ry'(\tilde{a}) > 0, \quad 0 < \tilde{a} < a_q,$$

so g_r is strictly increasing on $[b_r, a_q]$, and in particular on U_I.

Put $d_I = \operatorname{dist}(A_I, U_I^c) > 0$. If $rH < d_I/2$, then $a(y) \pm rH \in U_I$ for all $y \in I$. Since $g_r' \geq 1$ on U_I, the intermediate value theorem gives a unique solution to (65) in U_I, and

$$|\tilde{\alpha}_r(y, h) - a(y)| \leq rH.$$

The mean-value theorem gives, for some ξ_r between $\tilde{\alpha}_r(y, h)$ and $a(y)$,

$$\tilde{\alpha}_r(y, h) - a(y) = \frac{rh}{1 + ry'(\xi_r)}. \tag{68}$$

Since y' is bounded on $\overline{U_I}$ and $|h| \leq H$, this gives $\tilde{\alpha}_r(y, h) = a(y) + O(r)$, uniformly on I. Since y' is uniformly continuous on $\overline{U_I}$ and $|\tilde{\alpha}_r(y, h) - a(y)| \leq rH$, we have $y'(\xi_r) = y'\{a(y)\} + o(1)$ uniformly. Hence

$$\frac{\tilde{\alpha}_r(y, h)}{r} = \frac{a(y)}{r} + h - ry'\{a(y)\}h + o(r), \tag{69}$$

uniformly for $y \in I$ and $|h| \leq H$.

For a secondary non-null with index $\tilde{a} \in [b_r, a_q]$,

$$(\Theta_{\tilde{a}}, B_{\tilde{a}}) = \left(\frac{\tilde{a}}{r} + y(\tilde{a}), 1\right).$$

Conditional on $Z = -y$,

$$T_{\tilde{a}} = \frac{\tilde{a}}{r} + y(\tilde{a}) - y.$$

Uniformly for $y \in I$ and $\tilde{a} \in [b_r, a_q]$,

$$T_{\tilde{a}} \geq \frac{b_r}{r} + y(b_r) - \sup_{u \in I} u \longrightarrow \infty$$

by (47) in Lemma 2.9. Also $a(y)/r + h > 0$ uniformly for $y \in I$ and $|h| \leq H$. Therefore rejection at cutoff $a(y)/r + h$ is equivalent to

$$\frac{\tilde{a}}{r} + y(\tilde{a}) - y \geq \frac{a(y)}{r} + h,$$

or, equivalently,

$$g_r(\tilde{a}) \geq g_r\{a(y)\} + rh.$$

Since g_r is strictly increasing, a secondary non-null is rejected exactly when its index lies in $[\tilde{\alpha}_r(y, h), a_q]$.

Equations (68) and (69) give, for some $\eta_r(y, h)$ satisfying

$$\sup_{y \in I, \, |h| \leq H} \frac{|\eta_r(y, h)|}{r} \longrightarrow 0,$$

that

$$\frac{\tilde{\alpha}_r(y, h)}{r} = \frac{a(y)}{r} + h - ry'\{a(y)\}h + \eta_r(y, h).$$

Because $a(y)$ is bounded away from zero on I, $a(y)/r \to \infty$ uniformly. Applying (145) in Lemma A.2 to the two bounded shifts from $a(y)/r$ gives

$$\log \frac{p\{\tilde{\alpha}_r(y,h)/r\}}{p(a(y)/r+h)} = -\frac{a(y)}{r}\{-ry'\{a(y)\}h + \eta_r(y,h)\} \\ - \frac{(h - ry'\{a(y)\}h + \eta_r(y,h))^2 - h^2}{2} + o(1).$$

The first line equals $\kappa(y)h + o(1)$ uniformly, and the quadratic difference is $o(1)$ uniformly. This proves (67). $\square$

The following lemma identifies the analogue of Lemma 2.11.

Lemma 2.13. *For every compact $I \subset (1, y_q)$ and every $H > 0$, uniformly for $y \in I$ and $|h| \le H$,*

$$\bar{R}_{r,-y}\left(\frac{a(y)}{r} + h\right) \longrightarrow qM(y) + \{1 - qM(y)\}e^{\kappa(y)h}. \tag{70}$$

Moreover, for every $h_0 > 0$, there is $\gamma > 0$ such that, for all sufficiently small r,

$$\sup_{y\in I} \sup_{u_q \le c \le a(y)/r - h_0} \bar{R}_{r,-y}(c) \le 1 - \gamma. \tag{71}$$

Consequently, uniformly for $y \in I$,

$$C_r(-y) - \frac{a(y)}{r} \longrightarrow 0, \quad D_r(-y) \longrightarrow qM(y). \tag{72}$$

Moreover, for sufficiently small r, $C_r(-y)$ is regular for every $y \in I$.

Proof. **Proof of** (70)**.** Fix $H > 0$, take $y \in I$, and take $|h| \le H$. By Lemma 2.12, a secondary non-null with $a = \tilde{a}$ is rejected at cutoff $a(y)/r + h$ if and only if

$$\tilde{a} \in [\tilde{\alpha}_r(y,h), a_q].$$

By (41), their mass under the secondary component is

$$\tau_r \mu_r([\tilde{\alpha}_r(y,h), a_q]) = W_r\{\tilde{\alpha}_r(y,h)\}.$$

The definition (39) therefore gives

$$\frac{qW_r\{\tilde{\alpha}_r(y,h)\}}{p(a(y)/r+h)} = [1 - qM\{y(\tilde{\alpha}_r(y,h))\}]\frac{p\{\tilde{\alpha}_r(y,h)/r\}}{p(a(y)/r+h)}.$$

By (66), $y(\tilde{\alpha}_r(y,h)) \to y$ uniformly for $y \in I$ and $|h| \le H$. Equation (67) gives

$$\frac{p\{\tilde{\alpha}_r(y,h)/r\}}{p(a(y)/r+h)} \longrightarrow e^{\kappa(y)h},$$

uniformly for $y \in I$ and $|h| \le H$. Hence

$$\frac{qW_r\{\tilde{\alpha}_r(y,h)\}}{p(a(y)/r+h)} = [1 - qM\{y(\tilde{\alpha}_r(y,h))\}]\frac{p\{\tilde{\alpha}_r(y,h)/r\}}{p(a(y)/r+h)} \longrightarrow \{1 - qM(y)\}e^{\kappa(y)h} \tag{73}$$

The primary non-null is not rejected at cutoff $a(y)/r + h$. For the index i corresponding to it, $(\Theta_i, B_i) = (c_r + y(b_r), 1)$. Conditional on $Z = -y$,

$$T_i = c_r + y(b_r) - y.$$

Let $a_I = \inf_{u\in I} a(u) > 0$. For $|h| \le H$,

$$\sup_{y\in I} r|T_i| \le rc_r + r\sup_{y\in I}|y(b_r) - y| \longrightarrow 0, \quad \inf_{y\in I} r\left\{\frac{a(y)}{r} + h\right\} \ge a_I - rH \longrightarrow a_I > 0,$$

using (47) and (48) in Lemma 2.9. Hence $|T_i| < a(y)/r + h$ for sufficiently small r, uniformly over $y \in I$ and $|h| \le H$, so the primary non-null is not rejected at this cutoff.

Since $a(y)$ is bounded away from zero on I and $a(y) \le a_q$, for all sufficiently small r,

$$u_q \le \frac{a(y)}{r} + h \le \frac{a_q + 1}{r}$$

uniformly for $y \in I$ and $|h| \le H$. Thus (49) in Lemma 2.10, applied with $A = a_q + 1$ and $T = \sup_{u\in I} u$, gives

$$\bar{R}_{0,r,-y}(a(y)/r + h) \longrightarrow qL_{a(y)}(y) = qM(y). \tag{74}$$

Combining (73) and (74) proves (70).

Proof of (71)**.** Since $I \subset (1, y_q)$ is compact, $y_I := \inf I > 1$. By (47) in Lemma 2.9, $y(b_r) \downarrow 1$. Hence, with

$$\delta := \frac{y_I - 1}{2} > 0,$$

we have $y(b_r) \le 1 + \delta < y_I$ for sufficiently small r, and therefore

$$y - y(b_r) \ge y_I - y(b_r) \ge \delta, \quad y \in I.$$

Put

$$s_r(y) = c_r + y(b_r) - y.$$

Then

$$\delta \le y - y(b_r) \le \sup_{u\in I} u - y(b_r) < \infty, \quad s_r(y) \le c_r - \delta.$$

Conditional on $Z = -y$, the primary non-null has value $T_i = s_r(y)$, which is positive for sufficiently small r. Hence this primary non-null is rejected at cutoff s exactly when $s \le s_r(y)$. For $u_q \le s \le s_r(y)$, (44) and the bound by one on the secondary integral give

$$\bar{R}_{r,-y}(s) \le q\pi_{0,r}\frac{R_{0,r,-y}\{p(s)\}}{p(s)} + \frac{qr}{p(s)} + \frac{q\tau_r}{p(s)}.$$

Since I is compact, $T_I := \sup_{y\in I} y < \infty$. Also $s \le s_r(y) \le c_r - \delta$, so (48) gives

$$0 \le rs \le rc_r \longrightarrow 0$$

uniformly for $y \in I$ and $u_q \le s \le s_r(y)$. Thus, for any fixed $A > 0$, this cutoff range lies in $0 \le s \le A/r$ for all sufficiently small r. Applying (49) in Lemma 2.10 with $T = T_I$ gives

$$\frac{R_{0,r,-y}\{p(s)\}}{p(s)} = L_{rs}(y) + o(1)$$

uniformly for $y \in I$ and $u_q \le s \le s_r(y)$. Since $rs \to 0$ uniformly and $y \le T_I$,

$$L_{rs}(y) = \exp\left\{-\frac{(rs)^2}{2}\right\}\cosh(rsy) = 1 + o(1)$$

uniformly for $y \in I$ and $u_q \le s \le s_r(y)$. Since $\pi_{0,r} \to 1$, it follows that

$$q\pi_{0,r}\frac{R_{0,r,-y}\{p(s)\}}{p(s)} \le q + o(1).$$

Also p is decreasing, so $p(s) \geq p(s_r(y))$. Therefore, for any such cutoff,

$$\bar{R}_{r,-y}(s) \leq q + o(1) + \frac{q(r+\tau_r)}{p(s_r(y))}$$

uniformly for $y \in I$. By (43),

$$\frac{q(r+\tau_r)}{p(s_r(y))} = (1 - q\pi_{0,r})\frac{p(c_r)}{p\{c_r + y(b_r) - y\}}.$$

Since $c_r \to \infty$ by (48) and

$$\delta \leq y - y(b_r) \leq \sup_{u \in I} u - 1 =: \Delta_I^{\text{shift}} < \infty, \tag{75}$$

we have $s_r(y) = c_r + y(b_r) - y \to \infty$, uniformly for $y \in I$. Equation (75) shows that the shift $y - y(b_r)$ lies in the fixed compact interval $[\delta, \Delta_I^{\text{shift}}] \subset (0, \infty)$, so (147) in Lemma A.2 gives

$$\frac{p(c_r)}{p\{c_r + y(b_r) - y\}} = \frac{p\{s_r(y) + y - y(b_r)\}}{p(s_r(y))} \longrightarrow 0$$

uniformly for $y \in I$. Hence

$$\sup_{y \in I} \sup_{u_q \leq s \leq s_r(y)} \bar{R}_{r,-y}(s) \leq q + o(1) < 1. \tag{76}$$

Fix $h_0 > 0$. The complementary range $s_r(y) < s \leq a(y)/r - h_0$ is nonempty uniformly on I, because

$$\inf_{y \in I} \left\{ \frac{a(y)}{r} - h_0 - s_r(y) \right\} \geq \frac{\inf_{y \in I} a(y)}{r} - c_r - O_I(1) \longrightarrow \infty,$$

where $rc_r \to 0$ by (48) in Lemma 2.9. It remains to prove that there is $\gamma > 0$ such that, for all sufficiently small r,

$$\sup_{y \in I} \sup_{s_r(y) < s \leq a(y)/r - h_0} \bar{R}_{r,-y}(s) \leq 1 - \gamma \tag{77}$$

For $s > s_r(y)$, the identity $T_i = s_r(y) > 0$ gives $|T_i| = T_i < s$. For sufficiently small r, every secondary non-null value is positive uniformly over $y \in I$ and $\tilde{a} \in [b_r, a_q]$. On the range $s_r(y) < s \leq a(y)/r - h_0$, define

$$\beta_{r,y,s} := \inf\left\{ \tilde{a} \in [b_r, a_q] : \frac{\tilde{a}}{r} + y(\tilde{a}) - y \geq s \right\}. \tag{78}$$

The set is nonempty for sufficiently small r, since $\tilde{a} = a_q$ satisfies the inequality uniformly over $y \in I$. The map $\tilde{a} \mapsto \tilde{a}/r + y(\tilde{a}) - y$ is strictly increasing by Lemma A.1, so a secondary non-null with $a = \tilde{a}$ is rejected if and only if $\tilde{a} \in [\beta_{r,y,s}, a_q]$. Therefore, by (44),

$$\bar{R}_{r,-y}(s) = q\pi_{0,r}\frac{R_{0,r,-y}\{p(s)\}}{p(s)} + \frac{q\tau_r}{p(s)}\mu_r([\beta_{r,y,s}, a_q]). \tag{79}$$

Suppose (77) fails for this h_0. Then there are

$$r_n \downarrow 0, \quad y_n \in I, \quad s_{r_n}(y_n) < s_n \leq a(y_n)/r_n - h_0, \quad \liminf_{n \to \infty} \bar{R}_{r_n,-y_n}(s_n) \geq 1. \tag{80}$$

After taking a subsequence,

$$y_n \to y \in I, \quad a(y_n) \to a(y), \quad x_n = r_n s_n \to x \in [0, a(y)]. \tag{81}$$

Define

$$h_n := s_n - \frac{a(y_n)}{r_n}.$$

Then $h_n \le -h_0$ by (80). If $h_n = O(1)$, then after passing to a further subsequence $h_n \to h \le -h_0$. Equation (70) gives

$$\bar{R}_{r_n,-y_n}(s_n) \longrightarrow qM(y) + \{1 - qM(y)\}e^{\kappa(y)h} \le qM(y) + \{1 - qM(y)\}e^{-\kappa(y)h_0} < 1,$$

contradicting $\liminf_{n\to\infty} \bar{R}_{r_n,-y_n}(s_n) \ge 1$. Hence, after passing to a further subsequence if necessary, it remains to consider

$$h_n \to -\infty. \tag{82}$$

We next identify the rejected secondary non-nulls at cutoff s_n. A secondary non-null with index $\tilde{a} \in [b_{r_n}, a_q]$ has $(\Theta, B) = (\tilde{a}/r_n + y(\tilde{a}), 1)$. Conditional on $Z = -y_n$, its value is

$$T_{\tilde{a},n} = \frac{\tilde{a}}{r_n} + y(\tilde{a}) - y_n.$$

For all large n, this value is positive uniformly in $\tilde{a} \in [b_{r_n}, a_q]$, because $b_{r_n}/r_n \to \infty$ and $y(\tilde{a}) - y_n$ is bounded below. Hence $|T_{\tilde{a},n}| = T_{\tilde{a},n}$, and the rejection rule $|T_{\tilde{a},n}| \ge s_n$ is equivalent to

$$\frac{\tilde{a}}{r_n} + y(\tilde{a}) - y_n \ge s_n.$$

The rejected set is nonempty because the endpoint $\tilde{a} = a_q$ is rejected. Indeed, compactness of $I \subset (1, y_q)$ and continuity of $a(\cdot)$ give

$$\Delta_I := a_q - \sup_{u\in I} a(u) > 0.$$

Since $s_n \le a(y_n)/r_n - h_0$ by (80),

$$\frac{a_q}{r_n} + y_q - y_n - s_n \ge \frac{a_q - a(y_n)}{r_n} + y_q - y_n + h_0 > 0$$

for large n, because $a_q - a(y_n) \ge \Delta_I$ and $y_n \le \sup I < y_q$. Define

$$G_n(\tilde{a}) := g_{r_n}(\tilde{a}) - r_n(s_n + y_n), \quad \tilde{a} \in [b_{r_n}, a_q],$$

where g_r is defined in (64). Multiplying the rejection inequality by r_n, a secondary non-null with index $\tilde{a}$ is rejected if and only if

$$G_n(\tilde{a}) \ge 0.$$

By the monotonicity part of Lemma 2.12, G_n is continuous and strictly increasing on $[b_{r_n}, a_q]$ for all large n. Therefore a secondary non-null with $a = \tilde{a}$ is rejected if and only if $\tilde{a} \in [\beta_n, a_q]$, where

$$\beta_n := \inf\{\tilde{a} \in [b_{r_n}, a_q] : G_n(\tilde{a}) \ge 0\}. \tag{83}$$

Put

$$d_n := y_n - y(\beta_n). \tag{84}$$

If $\beta_n > b_{r_n}$, then $G_n(\beta_n) = 0$. Thus, in this case,

$$\frac{\beta_n}{r_n} - s_n = d_n. \tag{85}$$

If $\beta_n = b_{r_n}$, then $G_n(\beta_n) \ge 0$, so (85) is replaced by

$$\frac{\beta_n}{r_n} - s_n \ge d_n. \tag{86}$$

Moreover, $s_n \to \infty$, because $s_n > s_{r_n}(y_n) = c_{r_n} + y(b_{r_n}) - y_n$, $c_{r_n} \to \infty$, $y(b_{r_n}) \to 1$, and I is compact. Also $d_n > 0$ eventually. If $\beta_n = b_{r_n}$, this follows from the definition of d_n and (75), which give

$$d_n = y_n - y(b_{r_n}) \geq \delta > 0.$$

If $\beta_n > b_{r_n}$, then

$$g_{r_n}(\beta_n) = r_n(s_n + y_n) = g_{r_n}\{a(y_n)\} + r_n h_n \leq g_{r_n}\{a(y_n)\} - r_n h_0.$$

The strict monotonicity of g_{r_n} therefore gives $\beta_n < a(y_n)$, and hence $d_n = y_n - y(\beta_n) > 0$. The null term in (79) already has a limit along this subsequence:

$$q\pi_{0,r_n} \frac{R_{0,r_n,-y_n}\{p(s_n)\}}{p(s_n)} \longrightarrow qL_x(y) \leq qM(y) < 1. \tag{87}$$

Indeed, (49) in Lemma 2.10 applies because $r_n s_n \to x$, $y_n \to y$, and $\pi_{0,r_n} \to 1$. The inequality follows from $M(y) = \sup_{\alpha \geq 0} L_\alpha(y)$, the strict monotonicity of M on $(1, \infty)$, $y \in I \subset (1, y_q)$, and $qM(y_q) = 1$. Equations (79) and (87), together with the threshold characterization in (83), imply

$$\bar{R}_{r_n,-y_n}(s_n) = qL_x(y) + o(1) + \frac{q\tau_{r_n}}{p(s_n)} \mu_{r_n}([\beta_n, a_q]). \tag{88}$$

Moreover, (41), (39), (133) in Lemma A.1, and the bound from (85)–(86) give

$$\frac{q\tau_{r_n}}{p(s_n)} \mu_{r_n}([\beta_n, a_q]) = \frac{qW_{r_n}(\beta_n)}{p(s_n)} = [1 - qM\{y(\beta_n)\}] \frac{p(\beta_n/r_n)}{p(s_n)} \leq \frac{p(s_n + d_n)}{p(s_n)}. \tag{89}$$

It remains only to prove that

$$s_n d_n \longrightarrow \infty. \tag{90}$$

Equations (89), (90), and (148) in Lemma A.2 imply

$$\frac{q\tau_{r_n}}{p(s_n)} \mu_{r_n}([\beta_n, a_q]) = o(1).$$

Substituting this into (88) gives

$$\bar{R}_{r_n,-y_n}(s_n) \longrightarrow qL_x(y) < 1,$$

which yields a contradiction to (80).

To prove (90), we now consider three cases under (80), (81), and (82).

Case (a): $0 < x < a(y)$. Suppose $x > 0$. Since $b_{r_n} \to 0$ by (47) in Lemma 2.9 and $r_n s_n \to x$, the endpoint case $\beta_n = b_{r_n}$ cannot occur for large n: multiplying (86) by r_n would give

$$b_{r_n} - r_n s_n \geq r_n\{y_n - y(b_{r_n})\},$$

whose left side tends to $-x < 0$, while the right side is nonnegative for large n. Hence (85) holds eventually. Multiplying it by r_n gives

$$\beta_n - r_n s_n = r_n d_n. \tag{91}$$

The factor d_n is bounded. Indeed, I is compact, so $|y_n| \leq \sup_{u \in I} |u|$. Also $\beta_n \in [b_{r_n}, a_q]$, and $y(\cdot)$ is increasing by Lemma A.1; hence

$$y(b_{r_n}) \leq y(\beta_n) \leq y(a_q) = y_q. \tag{92}$$

Since $y(b_{r_n}) \downarrow 1$ by (47) in Lemma 2.9, the sequence $y(\beta_n)$ is bounded. Thus $r_n d_n \to 0$, so $\beta_n - r_n s_n \to 0$. Since $r_n s_n \to x$, we have $\beta_n \to x$. Since $x < a(y)$ and $y(\cdot)$ is strictly increasing by Lemma A.1, $y(x) < y$.

Together, $y_n \to y$ from (81) and $\beta_n \to x$ imply, by continuity of $y(\cdot)$, that $d_n \to d := y - y(x) > 0$. Also, (85) gives $\beta_n/r_n = s_n + d_n$. Therefore, for all large n,

$$\frac{\beta_n}{r_n} = s_n + d_n, \quad 0 < \frac{d}{2} \le d_n \le 2d, \quad s_n \to \infty.$$

As a result, (90) is proved.

Case (b): $x = 0$. Then $r_n s_n \to 0$. Since

$$s_n > s_{r_n}(y_n) = c_{r_n} + y(b_{r_n}) - y_n,$$

(48) gives $c_{r_n} \to \infty$, while (47) gives $y(b_{r_n}) \downarrow 1$ and compactness of I keeps y_n bounded. Hence $s_n \to \infty$. If $\beta_n = b_{r_n}$, then $\beta_n \to 0$ by (47). If $\beta_n > b_{r_n}$, then (85) holds; multiplying it by r_n, using compactness of I, and applying (92) gives

$$\beta_n - r_n s_n = r_n d_n \to 0.$$

Since $r_n s_n \to 0$, this also gives $\beta_n \to 0$. Thus $\beta_n \to 0$ in both cases. By (81), $y_n \to y$, and by the inverse relation together with (131) in Lemma A.1, $y(\beta_n) \to 1$. Hence

$$d_n \to y - 1 > 0.$$

Moreover, (85) in the interior case and (86) in the endpoint case give $\beta_n/r_n \ge s_n + d_n$. Since $d_n \to d := y - 1 > 0$ and $s_n \to \infty$, (90) is proved.

Case (c): $x = a(y)$. By (82), this is the only remaining subcase. Since $r_n s_n \to a(y)$ and $a(y_n) \to a(y)$, we have

$$r_n h_n = r_n s_n - a(y_n) \to a(y) - a(y) = 0.$$

The threshold equation $G_n(\tilde{a}) = 0$ is

$$g_{r_n}(\tilde{a}) = r_n(s_n + y_n) = a(y_n) + r_n y_n + r_n h_n = g_{r_n}\{a(y_n)\} + r_n h_n,$$

where $y\{a(y_n)\} = y_n$. Since $h_n < 0$ and g_{r_n} is strictly increasing by Lemma 2.12, the corresponding threshold lies below $a(y_n)$. To justify the existence and location of this threshold, choose $\eta > 0$ such that

$$[a(y_n) - \eta, a(y_n)] \subset (0, a_q)$$

for all large n; this is possible because $a(y_n) \to a(y) \in (0, a_q)$. Since $-r_n h_n \to 0$ and $g'_{r_n} \ge 1$, for all large n,

$$g_{r_n}\{a(y_n) - \eta\} < g_{r_n}\{a(y_n)\} + r_n h_n < g_{r_n}\{a(y_n)\}.$$

Continuity and strict monotonicity of g_{r_n} therefore give a unique $\tilde{\beta}_n \in (a(y_n) - \eta, a(y_n))$ satisfying

$$g_{r_n}(\tilde{\beta}_n) = g_{r_n}\{a(y_n)\} + r_n h_n.$$

Moreover, the mean-value theorem and $g'_{r_n} \ge 1$ give

$$0 < a(y_n) - \tilde{\beta}_n \le -r_n h_n \longrightarrow 0.$$

Because $a(y_n)$ remains bounded away from 0 and a_q, and $b_{r_n} \to 0$ by (47) in Lemma 2.9, it follows that

$$b_{r_n} < \tilde{\beta}_n < a_q$$

eventually. Thus $\tilde{\beta}_n$ is the threshold β_n in (83), so $G_n(\beta_n) = 0$. The mean-value theorem applied to g_{r_n} now gives, for some ξ_n between β_n and $a(y_n)$,

$$-r_n h_n = g_{r_n}\{a(y_n)\} - g_{r_n}(\beta_n) = (a(y_n) - \beta_n)\{1 + r_n y'(\xi_n)\}.$$

Therefore $\beta_n < a(y_n)$ and

$$a(y_n) - \beta_n = \frac{-r_n h_n}{1 + r_n y'(\xi_n)}. \tag{93}$$

For some ζ_n between β_n and $a(y_n)$,

$$\frac{\beta_n}{r_n} - s_n = d_n = y'(\zeta_n)\{a(y_n) - \beta_n\}. \tag{94}$$

Combining (93) and (94) gives

$$s_n d_n = (r_n s_n) \frac{y'(\zeta_n)}{1 + r_n y'(\xi_n)} (-h_n). \tag{95}$$

Here $r_n s_n \to a(y) > 0$. Also $\beta_n \to a(y)$ and $a(y_n) \to a(y)$, so $\xi_n, \zeta_n \to a(y)$; hence $y'(\zeta_n) \to y'\{a(y)\} > 0$, while $1 + r_n y'(\xi_n) \to 1$. Since $h_n \to -\infty$ by (82), (95) yields

$$s_n d_n \longrightarrow \infty.$$

Thus (90) is proved.

This proves (77). Combining (76) and (77), and reducing γ if necessary, proves (71).

Proof of (72). Fix $h > 0$. Equation (71) excludes a crossing before $a(y)/r - h$. Also,

$$qM(y) + \{1 - qM(y)\}e^{\kappa(y)h} > 1, \quad y \in I,$$

and the left side is bounded away from one on I, since I is compact, $qM(y) < 1$, and $\kappa(y) > 0$. Therefore (70), applied at h, gives a strict crossing by $a(y)/r + h$, uniformly on I. Hence

$$\frac{a(y)}{r} - h < C_r(-y) \le \frac{a(y)}{r} + h.$$

Let $h \downarrow 0$. Then

$$C_r(-y) - \frac{a(y)}{r} \longrightarrow 0, \quad rC_r(-y) \longrightarrow a(y),$$

uniformly on I.

We now verify regularity with r fixed. Fix $H = 1$, and, for $y \in I$ and $|h| \le H$, let $\tilde{\alpha}_r(y, h)$ be the threshold defined by (65). Write

$$\mathcal{N}_r(y, h) = q\pi_{0,r} \frac{R_{0,r,-y}\{p(a(y)/r + h)\}}{p(a(y)/r + h)},$$

$$\mathcal{S}_r(y, h) = \frac{qW_r\{\tilde{\alpha}_r(y, h)\}}{p(a(y)/r + h)} = \left[1 - qM\{y(\tilde{\alpha}_r(y, h))\}\right] \frac{p\{\tilde{\alpha}_r(y, h)/r\}}{p(a(y)/r + h)}.$$

The primary non-null is not rejected on this window. Indeed, for the corresponding index i, conditional on $Z = -y$,

$$T_i = c_r + y(b_r) - y.$$

Uniformly for $y \in I$ and $|h| \le H$,

$$r|T_i| \le rc_r + r \sup_{u \in I} |y(b_r) - u| \longrightarrow 0,$$

whereas

$$r\left\{\frac{a(y)}{r} + h\right\} = a(y) + rh \ge \inf_{u \in I} a(u) - rH$$

is bounded away from zero for all sufficiently small r. Hence $|T_i| < a(y)/r + h$ throughout the window, so the primary non-null is not rejected. By Lemma 2.12,

$$\bar{R}_{r,-y}\left(\frac{a(y)}{r} + h\right) = \mathcal{N}_r(y,h) + \mathcal{S}_r(y,h).$$

We first differentiate the secondary non-null contribution $\mathcal{S}_r(y,h)$. Equation (65) gives

$$\frac{\partial}{\partial h}\tilde{\alpha}_r(y,h) = \frac{r}{1 + ry'\{\tilde{\alpha}_r(y,h)\}}.$$

Put $\lambda(x) = \phi(x)/\overline{\Phi}(x)$. Since $-p'(x)/p(x) = \lambda(x)$, direct differentiation gives

$$\begin{aligned}\frac{\partial}{\partial h}\log \mathcal{S}_r(y,h) = &-\frac{qM'\{y(\tilde{\alpha}_r(y,h))\}y'\{\tilde{\alpha}_r(y,h)\}}{1 - qM\{y(\tilde{\alpha}_r(y,h))\}}\frac{r}{1 + ry'\{\tilde{\alpha}_r(y,h)\}} \\ &- \frac{\lambda\{\tilde{\alpha}_r(y,h)/r\}}{1 + ry'\{\tilde{\alpha}_r(y,h)\}} + \lambda\left(\frac{a(y)}{r} + h\right).\end{aligned}$$

The first term is $O(r)$ uniformly, because I is compact and $1 - qM\{y(\tilde{\alpha}_r(y,h))\}$ is bounded away from zero by (66) and (133) in Lemma A.1. Equations (66) and (69), together with (137) in Lemma A.2, give

$$\lambda\left(\frac{a(y)}{r} + h\right) - \frac{\lambda\{\tilde{\alpha}_r(y,h)/r\}}{1 + ry'\{\tilde{\alpha}_r(y,h)\}} = a(y)y'\{a(y)\} + o(1) = \kappa(y) + o(1)$$

uniformly for $y \in I$ and $|h| \le H$. Combining this with (73) yields

$$\frac{\partial}{\partial h}\mathcal{S}_r(y,h) \longrightarrow \kappa(y)\{1 - qM(y)\}e^{\kappa(y)h} \tag{96}$$

uniformly on the same set.

It remains to differentiate the null contribution. Put

$$x_\pm = \frac{a(y)/r + h \pm ry}{\sqrt{1 - r^2}}, \quad Q_\pm = \frac{\overline{\Phi}(x_\pm)}{\overline{\Phi}(a(y)/r + h)}.$$

Then

$$\frac{R_{0,r,-y}\{p(a(y)/r + h)\}}{p(a(y)/r + h)} = \frac{Q_+ + Q_-}{2},$$

and

$$\frac{\partial Q_\pm}{\partial h} = Q_\pm\left\{\lambda\left(\frac{a(y)}{r} + h\right) - \frac{\lambda(x_\pm)}{\sqrt{1 - r^2}}\right\}.$$

Since $a(y)$ is bounded away from zero on I and $a(y) \le a_q$, for all sufficiently small r,

$$u_q \le \frac{a(y)}{r} + h \le \frac{a_q + 1}{r}$$

uniformly for $y \in I$ and $|h| \le H$. Applying (49) in Lemma 2.10 with $A = a_q + 1$ and $T = \sup_{u \in I} u$ therefore shows that the quantities $Q_\pm$ are uniformly bounded. Moreover, (137) in Lemma A.2 gives

$$\lambda\left(\frac{a(y)}{r} + h\right) - \frac{\lambda(x_\pm)}{\sqrt{1 - r^2}} = \left(\frac{a(y)}{r} + h\right) - \frac{a(y)/r + h \pm ry}{1 - r^2} + O(r) = O(r)$$

uniformly. Therefore

$$\sup_{y \in I}\sup_{|h| \le H}\left|\frac{\partial}{\partial h}\mathcal{N}_r(y,h)\right| \longrightarrow 0. \tag{97}$$

Since I is compact and $\kappa(y)\{1-qM(y)\}e^{\kappa(y)h}$ is strictly positive on $I\times[-H,H]$, (96) and (97) imply

$$\inf_{y\in I}\inf_{|h|\le H}\frac{\partial}{\partial h}\bar{R}_{r,-y}\left(\frac{a(y)}{r}+h\right)>0$$

for all sufficiently small r.

After reducing γ if necessary, (71) with $h_0=H$ and (70) with $h=H$ give, uniformly for $y\in I$,

$$\sup_{u_q\le s\le a(y)/r-H}\bar{R}_{r,-y}(s)\le 1-\gamma,\quad \bar{R}_{r,-y}\left(\frac{a(y)}{r}+H\right)\ge 1+\gamma.$$

Thus,

$$C_r(-y)\in\left(\frac{a(y)}{r}-H,\frac{a(y)}{r}+H\right),$$

and the crossing at $C_r(-y)$ is unique within this local window. Fix such an r, a point $y\in I$, and any $0<\rho<C_r(-y)-u_q$. Split

$$[u_q,C_r(-y)-\rho]=A_-\cup A_0,$$

where

$$A_-=[u_q,C_r(-y)-\rho]\cap\left[u_q,\frac{a(y)}{r}-H\right],\quad A_0=[u_q,C_r(-y)-\rho]\cap\left[\frac{a(y)}{r}-H,\frac{a(y)}{r}+H\right].$$

On A_-, (71) gives $\bar{R}_{r,-y}(s)\le 1-\gamma$. On A_0, if it is nonempty, strict monotonicity on the local window gives

$$\sup_{s\in A_0}\bar{R}_{r,-y}(s)\le\bar{R}_{r,-y}\{C_r(-y)-\rho\}<\bar{R}_{r,-y}\{C_r(-y)\}=1.$$

This proves (16). For every $\rho>0$, strict monotonicity supplies a point

$$C_r(-y)<s<\min\left\{C_r(-y)+\rho,\frac{a(y)}{r}+H\right\}$$

at which $\bar{R}_{r,-y}(s)>1$, proving (17). The null contribution

$$s\longmapsto q\pi_{0,r}\frac{R_{0,r,-y}\{p(s)\}}{p(s)}$$

is smooth at the positive cutoff $C_r(-y)$, so $C_r(-y)$ is regular for each $y\in I$ and all sufficiently small r. Therefore

$$D_r(-y)=q\pi_{0,r}\frac{R_{0,r,-y}\{p(C_r(-y))\}}{p(C_r(-y))}\longrightarrow qL_{a(y)}(y)=qM(y).$$

Here the convergence follows from (49) in Lemma 2.10, $rC_r(-y)\to a(y)$, and $\pi_{0,r}\to 1$. □

2.6.4 The regime $z<-y_q$

Let $y=-z$. For $y>y_q$, let $\alpha_q(y)\in(0,a_q)$ be the unique solution of

$$qL_{\alpha_q(y)}(y)=1. \tag{98}$$

For fixed y, the map $a\mapsto L_a(y)$ is continuous and

$$\frac{\partial}{\partial a}\log L_a(y)=-a+y\tanh(ay). \tag{99}$$

By Lemma A.1, this derivative is positive on $(0, a(y))$. Since $y > y_q$, the same lemma gives $a(y) > a_q$, so $a \mapsto L_a(y)$ is strictly increasing on $[0, a_q]$. Moreover, by the definition of a_q in (36),

$$qL_0(y) = q < 1, \quad qL_{a_q}(y) > qL_{a_q}(y_q) = qM(y_q) = 1,$$

where the strict inequality follows from $a_q > 0$, $y > y_q$, and the monotonicity of $\cosh(a_q y)$ in $y > 0$. Thus continuity and strict monotonicity give exactly one solution to (98).

The next lemma analyzes the limiting conditional cutoff and FDP.

Lemma 2.14. *For every compact $I \subset (y_q, \infty)$, uniformly for $y \in I$,*

$$rC_r(-y) \longrightarrow \alpha_q(y), \quad D_r(-y) \longrightarrow 1. \tag{100}$$

Proof. Fix a compact $I \subset (y_q, \infty)$, and put

$$\delta_I = \inf_{y \in I}(y - y_q) > 0.$$

Let

$$s_r(y) = c_r + y(b_r) - y.$$

Recall from Lemma 2.12 that, for $\tilde{a} \in [b_r, a_q]$, $g_r(\tilde{a}) = \tilde{a} + ry(\tilde{a})$. By Lemma A.1, $g_r'(\tilde{a}) = 1 + ry'(\tilde{a}) > 0$. For $s > s_r(y)$, define

$$\mathcal{A}_{r,y,s} := \{\tilde{a} \in [b_r, a_q] : g_r(\tilde{a}) \geq r(s + y)\}, \qquad \beta_{r,y,s} := \begin{cases} \inf \mathcal{A}_{r,y,s}, & \mathcal{A}_{r,y,s} \neq \varnothing, \\ a_q, & \mathcal{A}_{r,y,s} = \varnothing. \end{cases} \tag{101}$$

For all sufficiently small r, the primary non-null is not rejected when $s > s_r(y)$. To verify the corresponding sign assertion for the secondary non-nulls, note that, conditional on $Z = -y$, the value associated with $\tilde{a} \in [b_r, a_q]$ is

$$\frac{\tilde{a}}{r} + y(\tilde{a}) - y \geq \frac{b_r}{r} + y(b_r) - \sup_{u \in I} u. \tag{102}$$

The monotonicity of $y(\cdot)$ gives (102). Its right-hand side tends to infinity as $r \downarrow 0$, by (47) in Lemma 2.9. Hence every secondary non-null value is positive for all sufficiently small r, uniformly over $y \in I$ and $\tilde{a} \in [b_r, a_q]$. Thus the rejected secondary non-nulls are indexed by $\mathcal{A}_{r,y,s}$. If this set is nonempty, continuity and strict monotonicity of g_r show that it is the closed interval $[\beta_{r,y,s}, a_q]$ and that

$$g_r(\beta_{r,y,s}) \geq r(s + y).$$

Consequently, in the nonempty case,

$$\frac{\beta_{r,y,s}}{r} - s \geq y - y(\beta_{r,y,s}) \geq y - y_q \geq \delta_I. \tag{103}$$

If $\mathcal{A}_{r,y,s} = \varnothing$, its μ_r-mass is zero. Since $\mu_r(\{a_q\}) = 0$ by (41), the convention in (101) gives in both cases

$$\mu_r(\mathcal{A}_{r,y,s}) = \mu_r([\beta_{r,y,s}, a_q]).$$

Using (41), (133) in Lemma A.1, and (103) when the set is nonempty, we obtain

$$0 \leq \frac{q\tau_r}{p(s)} \mu_r([\beta_{r,y,s}, a_q]) \leq \frac{p(s + \delta_I)}{p(s)}.$$

Moreover,

$$\inf_{y \in I} s_r(y) = c_r + y(b_r) - \sup_{y \in I} y \longrightarrow \infty,$$

because $c_r \to \infty$, $y(b_r) \to 1$, and I is compact. Therefore (147) in Lemma A.2 gives

$$\sup_{y\in I} \sup_{s>s_r(y)} \frac{q\tau_r}{p(s)} \mu_r([\beta_{r,y,s}, a_q]) \longrightarrow 0. \tag{104}$$

The function $(a, y) \mapsto L_a(y)$ is C^∞. Equation (99), together with $\alpha_q(y) < a_q < a(y)$ and Lemma A.1, shows that

$$\frac{\partial}{\partial a} \log L_a(y)\Big|_{a=\alpha_q(y)} > 0.$$

The implicit function theorem therefore shows that $y \mapsto \alpha_q(y)$ is continuous on (y_q, ∞). Since I is compact,

$$m_I := \inf_{y\in I} \min\{\alpha_q(y), a_q - \alpha_q(y), a(y) - \alpha_q(y)\} > 0. \tag{105}$$

Fix any $0 < \delta < m_I/2$. For every $y \in I$, continuity and strict monotonicity in a give

$$qL_{\alpha_q(y)-\delta}(y) < qL_{\alpha_q(y)}(y) = 1 < qL_{\alpha_q(y)+\delta}(y).$$

Both strict gaps are continuous functions of y. Their minima over the compact set I are positive, so there is $\gamma > 0$ such that, uniformly on I,

$$qL_{\alpha_q(y)-\delta}(y) \le 1 - 2\gamma, \quad qL_{\alpha_q(y)+\delta}(y) \ge 1 + 2\gamma \tag{106}$$

The choice $\delta < m_I/2$ also gives

$$\alpha_q(y) + \delta \le \alpha_q(y) + \frac{m_I}{2} \le a(y) - \frac{m_I}{2} < a(y), \tag{107}$$

uniformly for $y \in I$.

Write $g_y(x) = \log L_x(y)$. The proof of Lemma A.1, through (134), shows that $g_y'(x) > 0$ for $0 < x < a(y)$. By (107), every $x \in (0, \alpha_q(y) + \delta)$ lies in $(0, a(y))$. Hence $\partial_x L_x(y) = L_x(y) g_y'(x) > 0$ throughout $(0, \alpha_q(y) + \delta)$, and therefore $x \mapsto L_x(y)$ is increasing on $[0, \alpha_q(y) + \delta]$.

For $u_q \le s \le s_r(y)$, the population curve (44) and the fact that the primary and secondary non-null masses sum to $r + \tau_r$ give

$$\bar{R}_{r,-y}(s) \le q\pi_{0,r} \frac{R_{0,r,-y}\{p(s)\}}{p(s)} + \frac{q(r+\tau_r)}{p(s)}.$$

Since $s_r(y) = c_r + O_I(1)$,

$$0 \le rs \le rc_r + O_I(r) \longrightarrow 0$$

uniformly for $y \in I$ and $u_q \le s \le s_r(y)$. The set I is bounded, and this cutoff range lies in $[0, A/r]$ for some fixed $A > 0$ when r is sufficiently small. Hence (49) in Lemma 2.10 yields

$$\frac{R_{0,r,-y}\{p(s)\}}{p(s)} = L_{rs}(y) + o(1) = 1 + o(1)$$

uniformly for $y \in I$ and $u_q \le s \le s_r(y)$. Since $\pi_{0,r} \to 1$ and p is decreasing, it follows that

$$\bar{R}_{r,-y}(s) \le q + o(1) + \frac{q(r+\tau_r)}{p(s_r(y))}.$$

For the remaining term, (43) and $s_r(y) = c_r + y(b_r) - y$ give

$$\frac{q(r+\tau_r)}{p(s_r(y))} = (1 - q\pi_{0,r}) \frac{p\{s_r(y) + y - y(b_r)\}}{p(s_r(y))}.$$

Here $1 - q\pi_{0,r}$ is bounded. Also $s_r(y) \to \infty$ uniformly on I, and, because $I \subset (y_q, \infty)$ is compact while $y(b_r) \to 1$, the shifts $y - y(b_r)$ stay in a compact subinterval of $(0, \infty)$. Thus (147) in Lemma A.2 makes the ratio tend to zero uniformly on I. Thus,

$$\sup_{u_q \le s \le s_r(y)} \bar{R}_{r,-y}(s) \le q + o(1), \quad \text{uniformly for } y \in I.$$

For $s > s_r(y)$, the term in (44) that corresponds to the primary non-nulls is zero, and (104) and (49) in Lemma 2.10 give, uniformly for $y \in I$ and $s_r(y) < s \le (\alpha_q(y) + \delta)/r$,

$$\bar{R}_{r,-y}(s) = q\pi_{0,r} \frac{R_{0,r,-y}\{p(s)\}}{p(s)} + \frac{q\tau_r}{p(s)} \mu_r([\beta_{r,y,s}, a_q]) = qL_{rs}(y) + o(1).$$

Combining these two ranges gives

$$\sup_{u_q \le s \le (\alpha_q(y) - \delta)/r} \bar{R}_{r,-y}(s) \le qL_{\alpha_q(y)-\delta}(y) + o(1) \le 1 - \gamma. \tag{108}$$

Since $rs_r(y) \to 0$ uniformly on I and $\alpha_q(y) + \delta$ is bounded away from zero, we have

$$\frac{\alpha_q(y) + \delta}{r} > s_r(y)$$

for all sufficiently small r, uniformly on I. At the cutoff $(\alpha_q(y) + \delta)/r$, the term in (44) that corresponds to the primary non-nulls is zero, while the secondary term is nonnegative. Therefore (49) in Lemma 2.10 gives, uniformly for $y \in I$,

$$\bar{R}_{r,-y}\left(\frac{\alpha_q(y) + \delta}{r}\right) \ge q\pi_{0,r} \frac{R_{0,r,-y}\{p((\alpha_q(y) + \delta)/r)\}}{p((\alpha_q(y) + \delta)/r)} = qL_{\alpha_q(y)+\delta}(y) + o(1). \tag{109}$$

By (106), the right side is at least $1 + \gamma$ for all sufficiently small r. Hence

$$\alpha_q(y) - \delta < rC_r(-y) \le \alpha_q(y) + \delta.$$

Letting $\delta \downarrow 0$ gives

$$rC_r(-y) \longrightarrow \alpha_q(y),$$

uniformly for $y \in I$.

We now verify regularity for each fixed sufficiently small r. Fix one $0 < \delta < m_I/2$ as in (105) in Lemma 2.14 and consider the interval

$$\frac{\alpha_q(y) - \delta}{r} \le s \le \frac{\alpha_q(y) + \delta}{r}.$$

The primary non-null has conditional value

$$T_i = c_r + y(b_r) - y = s_r(y).$$

It is positive for sufficiently small r, uniformly in $y \in I$, and it is rejected at cutoff s exactly when $s \le s_r(y)$. Since $\delta < m_I/2$ and $m_I \le \alpha_q(y)$, we have $\alpha_q(y) - \delta \ge m_I/2$. On the other hand,

$$rs_r(y) = rc_r + r\{y(b_r) - y\} \longrightarrow 0$$

uniformly for $y \in I$, by (48), $y(b_r) \to 1$, and compactness of I. Therefore, for all sufficiently small r, for every $y \in I$ and every cutoff s satisfying

$$\frac{\alpha_q(y) - \delta}{r} \le s \le \frac{\alpha_q(y) + \delta}{r},$$

we have

$$s_r(y) < \frac{\alpha_q(y) - \delta}{r} \le s.$$

Thus every cutoff in this interval lies strictly to the right of the primary value $s_r(y)$, so the primary non-null is not rejected there. Moreover, uniformly for these y and s,

$$\begin{aligned} r(s+y) - g_r(b_r) &\ge \alpha_q(y) - \delta + r\{y - y(b_r)\} - b_r \ge \frac{m_I}{2} - o(1) > 0, \\ g_r(a_q) - r(s+y) &\ge a_q - \alpha_q(y) - \delta + r(y_q - y) \ge \frac{m_I}{2} - o(1) > 0. \end{aligned}$$

Strict monotonicity of g_r therefore implies that $\mathcal{A}_{r,y,s} \neq \varnothing$, that $\beta_{r,y,s} \in (b_r, a_q)$, and that

$$g_r(\beta_{r,y,s}) = r(s+y), \qquad \frac{\partial \beta_{r,y,s}}{\partial s} = \frac{r}{1 + r y'(\beta_{r,y,s})}.$$

The threshold equation also gives

$$\beta_{r,y,s} = rs + r\{y - y(\beta_{r,y,s})\} = rs + O(r)$$

uniformly. Hence, for all sufficiently small r,

$$\frac{m_I}{4} \le \beta_{r,y,s} \le a_q - \frac{m_I}{4},$$

so the thresholds remain in a fixed compact subinterval of $(0, a_q)$.

For $y \in I$ and cutoffs s in the local crossing interval

$$\frac{\alpha_q(y) - \delta}{r} \le s \le \frac{\alpha_q(y) + \delta}{r},$$

write the null and secondary non-null contributions as

$$\mathcal{N}_{r,y}(s) = q\pi_{0,r} \frac{R_{0,r,-y}\{p(s)\}}{p(s)}, \quad \mathcal{S}_{r,y}(s) = [1 - qM\{y(\beta_{r,y,s})\}] \frac{p(\beta_{r,y,s}/r)}{p(s)}.$$

Then $\bar{R}_{r,-y}(s) = \mathcal{N}_{r,y}(s) + \mathcal{S}_{r,y}(s)$. Put $\lambda(x) = \phi(x)/\overline{\Phi}(x)$, and abbreviate $\beta = \beta_{r,y,s}$. Since

$$\mathcal{S}_{r,y}(s) = [1 - qM\{y(\beta)\}] \frac{p(\beta/r)}{p(s)}, \quad \frac{\partial \beta}{\partial s} = \frac{r}{1 + r y'(\beta)},$$

the chain rule first gives

$$\begin{aligned} \frac{\partial}{\partial s} \log \mathcal{S}_{r,y}(s) = &\frac{-qM'\{y(\beta)\} y'(\beta)}{1 - qM\{y(\beta)\}} \frac{\partial \beta}{\partial s} \\ &+ \frac{p'(\beta/r)}{p(\beta/r)} \frac{1}{r} \frac{\partial \beta}{\partial s} - \frac{p'(s)}{p(s)}. \end{aligned}$$

Using $-p'(x)/p(x) = \lambda(x)$, this becomes

$$\begin{aligned} \frac{\partial}{\partial s} \log \mathcal{S}_{r,y}(s) = &- \frac{qM'\{y(\beta)\} y'(\beta)}{1 - qM\{y(\beta)\}} \frac{r}{1 + r y'(\beta)} \\ &- \frac{\lambda(\beta/r)}{1 + r y'(\beta)} + \lambda(s). \end{aligned} \tag{110}$$

The compact containment of β in $(0, a_q)$, together with (133) in Lemma A.1, implies

$$\inf_{\substack{y \in I \\ |rs - \alpha_q(y)| \le \delta}} [1 - qM\{y(\beta)\}] > 0, \qquad \sup_{\substack{y \in I \\ |rs - \alpha_q(y)| \le \delta}} (|M'\{y(\beta)\}| + y'(\beta)) < \infty.$$

Thus the first term on the right side of (110) is $O(r)$ uniformly. For the remaining two terms, (137) in Lemma A.2 and $s \asymp \beta/r \asymp r^{-1}$ give

$$\lambda(s) = s + O(r), \quad \lambda(\beta/r) = \frac{\beta}{r} + O(r)$$

uniformly. The threshold equation $g_r(\beta) = r(s+y)$ is equivalent to

$$\frac{\beta}{r} = s + y - y(\beta).$$

Consequently,

$$\begin{aligned}\lambda(s) - \frac{\lambda(\beta/r)}{1+ry'(\beta)} &= s - \frac{\beta/r}{1+ry'(\beta)} + O(r) \\ &= \frac{rsy'(\beta) + y(\beta) - y}{1+ry'(\beta)} + O(r) = O(1),\end{aligned}$$

because rs, y, $y(\beta)$, and $y'(\beta)$ are uniformly bounded. Hence

$$\sup_{y\in I} \sup_{|rs-\alpha_q(y)|\le\delta} \left| \frac{\partial}{\partial s} \log \mathcal{S}_{r,y}(s) \right| = O(1). \tag{111}$$

Moreover, $\beta_{r,y,s}/r - s \ge \delta_I$ by (103), so $\beta_{r,y,s}/r \ge s + \delta_I$. Since p is decreasing on $[0,\infty)$, and since (133) gives $0 \le 1 - qM\{y(\beta_{r,y,s})\} \le 1$, we have

$$0 \le \mathcal{S}_{r,y}(s) \le \frac{p(s+\delta_I)}{p(s)}.$$

Applying (144) in Lemma A.2 with $x = s$ and $\delta = \delta_I$ gives

$$0 \le \mathcal{S}_{r,y}(s) \le \frac{p(s+\delta_I)}{p(s)} \le (1+s^{-2}) \exp\left\{-s\delta_I - \frac{\delta_I^2}{2}\right\} = o(r), \tag{112}$$

uniformly on the interval. The last $o(r)$ follows because (105) in Lemma 2.14 and $0 < \delta < m_I/2$ imply

$$s \ge \frac{\alpha_q(y) - \delta}{r} \ge \frac{m_I}{2r}$$

whenever $y \in I$ and $|rs - \alpha_q(y)| \le \delta$, and therefore

$$r^{-1}(1+s^{-2}) \exp\left\{-s\delta_I - \frac{\delta_I^2}{2}\right\} \le r^{-1}(1+o(1)) \exp\left\{-\frac{m_I\delta_I}{2r} - \frac{\delta_I^2}{2}\right\} \longrightarrow 0$$

uniformly. Since $\partial_s \mathcal{S}_{r,y}(s) = \mathcal{S}_{r,y}(s)\partial_s \log \mathcal{S}_{r,y}(s)$, (111) and (112) imply

$$\sup_{y\in I} \sup_{|rs-\alpha_q(y)|\le\delta} \left| \frac{\partial}{\partial s} \mathcal{S}_{r,y}(s) \right| = o(r). \tag{113}$$

For the null term, put

$$x_\pm = \frac{s \pm ry}{\sqrt{1-r^2}}, \quad Q_\pm = \frac{\overline{\Phi}(x_\pm)}{\overline{\Phi}(s)}.$$

Then

$$\frac{R_{0,r,-y}\{p(s)\}}{p(s)} = \frac{Q_+ + Q_-}{2}, \quad \frac{\partial Q_\pm}{\partial s} = Q_\pm \left\{\lambda(s) - \frac{\lambda(x_\pm)}{\sqrt{1-r^2}}\right\}.$$

Equation (50) in Lemma 2.10, together with (138) in Lemma A.2, gives uniformly for $y \in I$ and $|rs - \alpha_q(y)| \le \delta$

$$Q_\pm = \exp\left\{-\frac{(rs)^2}{2} \mp rsy\right\} + o(1), \qquad \frac{1}{r}\left\{\lambda(s) - \frac{\lambda(x_\pm)}{\sqrt{1-r^2}}\right\} = -rs \mp y + o(1).$$

Therefore

$$\frac{1}{r}\frac{\partial}{\partial s}\mathcal{N}_{r,y}(s) = qe^{-(rs)^2/2}\{-rs\cosh(rsy) + y\sinh(rsy)\} + o(1) = q\frac{\partial}{\partial a}L_a(y)\bigg|_{a=rs} + o(1). \tag{114}$$

By (105) and (107) in Lemma 2.14, the pairs (rs, y) under consideration range over a compact subset of $\{(a,y) : y \in I,\ 0 < a < a(y)\}$. Continuity, (99), and Lemma A.1 therefore show that $\partial_a L_a(y)|_{a=rs}$ is bounded away from zero uniformly. Combining (113) and (114) yields

$$\inf_{y\in I}\inf_{|rs-\alpha_q(y)|\le\delta}\frac{\partial}{\partial s}\bar{R}_{r,-y}(s) > 0$$

for all sufficiently small r.

Thus $s \mapsto \bar{R}_{r,-y}(s)$ is continuous and strictly increasing on $[\{\alpha_q(y)-\delta\}/r, \{\alpha_q(y)+\delta\}/r]$. Equation (108) gives

$$\bar{R}_{r,-y}(s) \le 1-\gamma, \qquad u_q \le s \le \frac{\alpha_q(y)-\delta}{r},$$

while (109) puts it strictly above one at $(\alpha_q(y)+\delta)/r$. Hence $C_r(-y)$ is the unique point in this interval at which $\bar{R}_{r,-y}\{C_r(-y)\} = 1$.

Fix $0 < \rho < C_r(-y) - u_q$, and write

$$L_y = \frac{\alpha_q(y)-\delta}{r}, \qquad U_y = \frac{\alpha_q(y)+\delta}{r}.$$

On the part of $[u_q, C_r(-y)-\rho]$ below the local interval, namely $[u_q, C_r(-y)-\rho] \cap [u_q, L_y]$, Equation (108) gives $\bar{R}_{r,-y}(s) \le 1-\gamma$. On the part inside the local interval, if $[u_q, C_r(-y)-\rho] \cap [L_y, U_y]$ is nonempty, then strict monotonicity of $s \mapsto \bar{R}_{r,-y}(s)$ on $[L_y, U_y]$ gives

$$\sup_{L_y \le s \le C_r(-y)-\rho}\bar{R}_{r,-y}(s) = \bar{R}_{r,-y}\{C_r(-y)-\rho\} < \bar{R}_{r,-y}\{C_r(-y)\} = 1.$$

Combining the two parts proves (16). For every $\rho > 0$, choose

$$C_r(-y) < s < \min\left\{C_r(-y)+\rho, \frac{\alpha_q(y)+\delta}{r}\right\}.$$

Strict monotonicity gives $\bar{R}_{r,-y}(s) > 1$, proving (17). Finally, the null curve

$$s \longmapsto q\pi_{0,r}\frac{R_{0,r,-y}\{p(s)\}}{p(s)}$$

is smooth at the positive cutoff $C_r(-y)$. Hence the crossing is regular in the sense of Definition 2.4. Therefore (46) gives

$$D_r(-y) = q\pi_{0,r}\frac{R_{0,r,-y}\{p(C_r(-y))\}}{p(C_r(-y))}.$$

Since $rC_r(-y) \to \alpha_q(y)$ and $\pi_{0,r} \to 1$, (49) in Lemma 2.10 yields

$$D_r(-y) \longrightarrow qL_{\alpha_q(y)}(y).$$

Finally, (98) is exactly $qL_{\alpha_q(y)}(y) = 1$, so $D_r(-y) \to 1$. $\square$

2.6.5 Conditional limit of D_r

The following lemma establishes the limit of $D_r(z)$ as $r \downarrow 0$, with D_* defined in Section 2.2.

Lemma 2.15. *For every $z \notin \{-1, -y_q\}$,*

$$D_r(z) \longrightarrow D_*(z). \tag{115}$$

Consequently,

$$\mathbb{E}[D_r(Z)] \longrightarrow \mathbb{E}[D_*(Z)] = \ell(q). \tag{116}$$

Proof. If $z > -1$, Lemma 2.11 gives $D_r(z) \to q$. If $-y_q < z < -1$, put $y = -z$ and apply Lemma 2.13. If $z < -y_q$, apply Lemma 2.14. This proves (115).

At every regular crossing, $0 \le D_r \le 1$; the value assigned at nonregular points also lies in $[0, 1]$. Dominated convergence gives the convergence in (116). Finally,

$$\begin{aligned}
\mathbb{E}[D_*(Z)] &= q\mathbb{P}(Z > -1) + q\int_{-y_q}^{-1} M(-z)\phi(z)\,\mathrm{d}z + \mathbb{P}(Z < -y_q) \\
&= q\Phi(1) + q\int_{1}^{y_q} M(y)\phi(y)\,\mathrm{d}y + \overline{\Phi}(y_q) \\
&= \ell(q).
\end{aligned}$$

□

2.7 Proofs of Theorem 2.1 and Propositions 2.2 and 2.3

Proof of Theorem 2.1. Fix $\delta > 0$. Choose a compact set

$$E \subset \mathbb{R} \setminus \{-1, -y_q\}$$

such that

$$\int_E D_*(z)\phi(z)\,\mathrm{d}z > \ell(q) - \frac{\delta}{4}.$$

The proofs of Lemmas 2.11, 2.13, and 2.14 give, for all sufficiently small r, brackets

$$A_r(z) < B_r(z), \quad z \in E,$$

and a constant $\gamma > 0$, such that

$$\sup_{z\in E}\sup_{u_q\le c\le A_r(z)} \bar{R}_{r,z}(c) \le 1-\gamma, \quad \inf_{z\in E} \bar{R}_{r,z}\{B_r(z)\} \ge 1+\gamma,$$

$$\sup_{z\in E}\sup_{A_r(z)\le c\le B_r(z)} |\bar{R}_{0,r,z}(c) - D_*(z)| \le \frac{\delta}{8}.$$

Fix such an r and put $C_* = \sup_{z\in E} B_r(z) < \infty$. The DKW argument used in Theorem 2.7, with these brackets in place of the regularity brackets, implies that for all sufficiently large N,

$$\mathbb{E}_{\Omega_N}[\mathrm{FDR}_{\Omega_N}(\mathrm{BH}_q)] > \ell(q) - \frac{\delta}{2}.$$

For such an N, some realization $\omega^\star$ satisfies

$$\mathrm{FDR}_{\omega^\star}(\mathrm{BH}_q) > \ell(q) - \frac{\delta}{2}.$$

By Lemma 2.8, choose $\eta > 0$ so small that

$$\mathrm{FDR}^{(\eta)}_{\omega^\star}(\mathrm{BH}_q) > \mathrm{FDR}_{\omega^\star}(\mathrm{BH}_q) - \frac{\delta}{2} > \ell(q) - \delta. \tag{117}$$

The perturbed vector $T^{(\eta)}$ has finite dimension N, mean vector $(\theta_i)_{i=1}^N \in \mathbb{R}^N$, and covariance matrix $\Sigma_{\omega^\star,\eta}$, which has unit diagonal and is strictly positive definite by (32) in Lemma 2.8. Thus the bound (117) establishes (5). Since $\delta > 0$ was arbitrary, taking the supremum proves (6). □

Proof of Proposition 2.2. Recall from Lemma A.1 that, for $y > 1$, $a(y)$ is the unique positive maximizer of $a \mapsto L_a(y)$, equivalently the unique positive solution of

$$a(y) = y \tanh\{a(y)y\}.$$

By the monotonicity of $a(\cdot)$ established in that lemma, $a(y) \geq a(2) > 0$ for $y \geq 2$, so $a(y)y \to \infty$ and

$$\frac{a(y)}{y} = \tanh\{a(y)y\} \longrightarrow 1.$$

Moreover,

$$0 \leq y - a(y) = y\,[1 - \tanh\{a(y)y\}] \leq 2ye^{-2a(y)y} \longrightarrow 0.$$

Because $a(y)$ is the maximizer defining $M(y)$, we have $M(y) = L_{a(y)}(y)$. Substituting the definitions of $L_{a(y)}(y)$ and $\phi(y)$, expanding $\cosh(t) = (e^t + e^{-t})/2$, and completing the two squares gives

$$\begin{aligned} M(y)\phi(y) &= e^{-a(y)^2/2}\cosh\{a(y)y\}\frac{e^{-y^2/2}}{\sqrt{2\pi}} \\ &= \frac{1}{2\sqrt{2\pi}}\left[e^{-\{a(y)^2+y^2\}/2+a(y)y} + e^{-\{a(y)^2+y^2\}/2-a(y)y}\right] \\ &= \frac{1}{2\sqrt{2\pi}}\left[e^{-\{y-a(y)\}^2/2} + e^{-\{y+a(y)\}^2/2}\right]. \end{aligned} \tag{118}$$

Consequently, $M(y)\phi(y) \to 1/(2\sqrt{2\pi})$. The difference $M(y)\phi(y) - 1/(2\sqrt{2\pi})$ is integrable on $[1,\infty)$. Indeed, for all sufficiently large y, $a(y) \geq y/2$, so

$$y - a(y) \leq 2ye^{-y^2},$$

while

$$\left|e^{-\{y-a(y)\}^2/2} - 1\right| \leq \frac{\{y-a(y)\}^2}{2}, \quad e^{-\{y+a(y)\}^2/2} \leq e^{-y^2/2}.$$

It follows from the definition of c_ℓ that, as $Y \to \infty$,

$$\int_1^Y M(y)\phi(y)\,\mathrm{d}y = \frac{Y}{2\sqrt{2\pi}} + c_\ell - \Phi(1) + o(1). \tag{119}$$

Next, (118) gives

$$M(y) = \frac{1}{2}e^{y^2/2}\left[e^{-\{y-a(y)\}^2/2} + e^{-\{y+a(y)\}^2/2}\right] = \frac{1}{2}e^{y^2/2}\{1 + o(1)\}.$$

Since $M(y_q) = 1/q \to \infty$, (131) in Lemma A.1 implies $y_q \to \infty$. The identity $qM(y_q) = 1$ therefore gives

$$y_q^2 = 2\log(1/q) + 2\log 2 + o(1), \quad y_q = \sqrt{2\log(1/q)} + o(1). \tag{120}$$

Equations (149) in Lemma A.2 and (120) give

$$\overline{\Phi}(y_q) \le \frac{\phi(y_q)}{y_q} = O\left(\frac{q}{y_q}\right) = o(q). \tag{121}$$

Substituting (119), (120), and (121) into (4) yields

$$\begin{aligned}\ell(q) &= q\Phi(1) + q\left\{\frac{y_q}{2\sqrt{2\pi}} + c_\ell - \Phi(1) + o(1)\right\} + o(q)\\ &= \frac{q\sqrt{\log(1/q)}}{2\sqrt{\pi}} + c_\ell q + o(q).\end{aligned}$$

Numerical quadrature of the absolutely convergent integral defining c_ℓ gives $c_\ell = 0.6492828\ldots$. □

Proof of Proposition 2.3. Since $q \in (0,1)$, (3) gives $M(y_q) = 1/q > 1$. By Lemma A.1, $M = 1$ on $[0,1]$ and M is strictly increasing on $(1,\infty)$, so $y_q > 1$ and $M(y) \ge 1$ for $1 \le y \le y_q$. Using (4),

$$\begin{aligned}\ell(q) - q &= q\Phi(1) + q\int_1^{y_q} M(y)\phi(y)\,\mathrm{d}y + \overline{\Phi}(y_q)\\ &\quad - q\left\{\Phi(1) + \int_1^{y_q}\phi(y)\,\mathrm{d}y + \overline{\Phi}(y_q)\right\}\\ &= q\int_1^{y_q}\{M(y)-1\}\phi(y)\,\mathrm{d}y + (1-q)\overline{\Phi}(y_q).\end{aligned}$$

The integral is nonnegative, and $(1-q)\overline{\Phi}(y_q) > 0$ because $q < 1$ and $y_q < \infty$. Hence $\ell(q) > q$. □

3 Upper bound for a class of common-factor models

In this section, we examine a class of common-factor models that contains the construction in Subsection 2.5. We show that the upper bound is $O(q\sqrt{\log(1/q)})$, which gives matching orders for small q.

Theorem 3.1. *Let*

$$T_i = \mu_i + a_i Z + \sqrt{1-a_i^2}\,\varepsilon_i, \quad 1 \le i \le m,$$

where $Z, \varepsilon_1, \ldots, \varepsilon_m$ are i.i.d. standard normal random variables and $|a_i| \le 1$. Form the two-sided Gaussian p-values $P_i = p(|T_i|)$ and apply BH at level $0 < q \le 2\overline{\Phi}(1) \approx 0.3173$. Let $\mathcal{H}_0 = \{i : \mu_i = 0\}$, $m_0 = |\mathcal{H}_0|$, and $\pi_0 = m_0/m$. If u_q is defined by $2\overline{\Phi}(u_q) = q$, then $\mathrm{FDR}(\mathrm{BH}_q) = 0$ *when $m_0 = 0$, and otherwise*

$$\mathrm{FDR}(\mathrm{BH}_q) \le q\left\{1 + \pi_0\left(\frac{1-q}{2} + \frac{u_q}{\sqrt{2\pi}}\right)\right\}.$$

Consequently, the worst-case FDR in this common-factor class is $O\{q\sqrt{\log(1/q)}\}$, uniformly in m, the means $(\mu_1, \ldots, \mu_m)$, and the loadings $(a_1, \ldots, a_m)$, as $q \downarrow 0$.

Remark 3.2 (PRDN for the two-sided null p-values)*.* The theorem imposes no sign restriction on the loadings a_i, so the null correlations $a_i a_j$ may be negative. Nevertheless, the two-sided null p-values satisfy PRDN. Let $T_0 = (T_i : i \in \mathcal{H}_0)$ be the null subvector, let $a_0 = (a_i : i \in \mathcal{H}_0)$, and put

$$D_0 = \mathrm{diag}(1 - a_i^2 : i \in \mathcal{H}_0).$$

The null covariance matrix is

$$\Sigma_0 = D_0 + a_0 a_0^\top.$$

Assume first that $|a_i| < 1$ for every null coordinate. Let

$$B = \operatorname{diag}\{\operatorname{sign}(a_i) : i \in \mathcal{H}_0\}, \quad r = |a_0|,$$

with $\operatorname{sign}(0) = 1$. Then

$$B\Sigma_0 B = D_0 + rr^\top.$$

By the Sherman–Morrison formula,

$$(D_0 + rr^\top)^{-1} = D_0^{-1} - \frac{D_0^{-1} rr^\top D_0^{-1}}{1 + r^\top D_0^{-1} r}.$$

Thus every off-diagonal entry of $(D_0+rr^\top)^{-1}$ is nonpositive. The Karlin–Rinott absolute-value multinormal MTP_2 criterion [Karlin and Rinott, 1981] therefore applies to BT_0, and $(|(BT_0)_i| : i \in \mathcal{H}_0)$ is MTP_2. Since $|(BT_0)_i| = |T_i|$, coordinatewise sign changes do not alter the two-sided p-values $p(|T_i|)$.

If some $|a_i| = 1$, define $a_i^{(\eta)} = (1-\eta)a_i$ for all $i \in \mathcal{H}_0$, and let $\eta \downarrow 0$. The corresponding absolute null vectors converge weakly to $(|T_i| : i \in \mathcal{H}_0)$. Under the measure-theoretic definition of MTP_2, the class of MTP_2 probability measures is closed under weak convergence [Colangelo et al., 2006]. Hence the limiting absolute-value law remains MTP_2. Appendix A.1 of Su [2018] then implies that the induced two-sided null p-values satisfy PRDN.

Proof. A leave-one-out form of BH self-consistency is standard in FDR proofs [Benjamini and Yekutieli, 2001]; we include the argument for completeness. If $m_0 = 0$, then $V = 0$ identically and hence $\mathrm{FDR} = 0$. Assume henceforth that $m_0 \geq 1$. Fix a true null i. Let R_i be the BH rejection count after replacing its p-value by zero; in particular, $R_i \geq 1$. Pointwise in the p-value vector,

$$\frac{\mathbf{1}\{i \text{ is rejected}\}}{R \vee 1} = \frac{\mathbf{1}\{P_i \leq qR_i/m\}}{R_i}.$$

Indeed, if i is rejected, replacing P_i by zero does not change the step-up rejection count. Conversely, if $P_i \leq qR_i/m$, restoring P_i leaves at least R_i values at or below qR_i/m. Thus at least R_i values remain below the BH threshold qR_i/m, so BH makes at least R_i rejections. Coordinatewise monotonicity gives at most R_i rejections. Here ties cause no ambiguity: the BH rejection set is $\{j : P_j \leq qR/m\}$ and has exactly R elements. If more than R values were at most qR/m, then $P_{(R+1)} \leq qR/m < q(R+1)/m$, contradicting maximality of R. Summing gives the leave-one-out identity

$$\mathrm{FDR} = \sum_{i \in \mathcal{H}_0} \mathbb{E}\left[\frac{\mathbf{1}\{P_i \leq qR_i/m\}}{R_i}\right]. \tag{122}$$

Conditional on $Z = z$, P_i is independent of R_i. Indeed, set $s_j = \sqrt{1 - a_j^2}$ and let $\mathcal{R}(\cdot)$ denote the BH step-up rejection count. Then

$$R_i = \mathcal{R}(P_1, \ldots, P_{i-1}, 0, P_{i+1}, \ldots, P_m), \quad P_j = p(|\mu_j + a_j z + s_j \varepsilon_j|), \quad j \neq i.$$

Hence, conditional on $Z = z$, R_i is measurable with respect to $\sigma(\varepsilon_j : j \neq i,\ s_j > 0)$; coordinates with $s_j = 0$ are constants after conditioning. Meanwhile $P_i = p(|a_i z + s_i \varepsilon_i|)$ for a true null i, or is deterministic if $s_i = 0$. Since ε_i is independent of $\{\varepsilon_j : j \neq i\}$, the claim follows. With

$$F_{i,z}(t) = \mathbb{P}(P_i \leq t \mid Z = z),$$

and $0 < qR_i/m \le q$,

$$\begin{aligned}
\mathbb{E}\left[\frac{\mathbf{1}\{P_i \le qR_i/m\}}{R_i}\,\middle|\, Z=z\right] &= \mathbb{E}\left[\frac{1}{R_i}\mathbb{P}\left(P_i \le \frac{qR_i}{m} \mid Z=z, R_i\right)\middle|\, Z=z\right] \\
&= \mathbb{E}\left[\frac{F_{i,z}(qR_i/m)}{R_i}\,\middle|\, Z=z\right] \\
&= \frac{q}{m}\,\mathbb{E}\left[\frac{F_{i,z}(qR_i/m)}{qR_i/m}\,\middle|\, Z=z\right] \\
&\le \frac{q}{m}\sup_{0<t\le q}\frac{F_{i,z}(t)}{t}.
\end{aligned} \tag{123}$$

Summing (123) over the true nulls gives

$$\mathbb{E}(\mathrm{FDP} \mid Z=z) \le \frac{q}{m}\sum_{i\in\mathcal{H}_0}\sup_{0<t\le q}\frac{F_{i,z}(t)}{t}. \tag{124}$$

To average (124) over Z, it remains to control

$$\int_{-u_q}^{u_q}\phi(z)\sup_{0<t\le q}\frac{F_{i,z}(t)}{t}\,dz$$

for each true null i. We do this on $|z|\le u_q$; the event $|Z|>u_q$ is handled at the end by FDP ≤ 1. Fix a true null index i. Put $r=|a_i|$ and $s=\sqrt{1-r^2}$, and first suppose $s>0$. Conditional on $Z=z$, $T_i \sim N(a_i z, s^2)$. The inequality $t=p(u)=2\overline{\Phi}(u)\le q$ holds if and only if $u\ge u_q$, and in this case

$$F_{i,z}(p(u)) = \int_u^\infty s^{-1}\phi\{(x-a_iz)/s\}\,dx + \int_u^\infty s^{-1}\phi\{(x+a_iz)/s\}\,dx.$$

Since $s^2 = 1-a_i^2$,

$$\begin{aligned}
\frac{s^{-1}\phi\{(x-a_iz)/s\}}{\phi(x)} &= \frac{1}{s}\exp\left\{-\frac{(x-a_iz)^2}{2s^2}+\frac{x^2}{2}\right\} \\
&= \frac{1}{s}\exp\left\{\frac{z^2}{2}-\frac{(a_ix-z)^2}{2s^2}\right\}.
\end{aligned}$$

The second tail is the same calculation with z replaced by $-z$. Choose $\eta_+,\eta_-\in\{-1,1\}$ with $\{\eta_+,\eta_-\}=\{-1,1\}$ so that $\eta_+a_iz\ge 0$ and $\eta_-a_iz\le 0$. For $x\ge u$,

$$|a_ix-\eta_+z| \ge (ru-|z|)_+, \qquad |a_ix-\eta_-z|\ge ru+|z|.$$

Both right-hand sides are nondecreasing in u, so for $u\ge u_q$,

$$\begin{aligned}
\phi(z)\frac{F_{i,z}(p(u))}{p(u)} &\le \frac{\phi(z)}{p(u)}\frac{e^{z^2/2}}{s}\left[e^{-(ru_q-|z|)_+^2/(2s^2)}+e^{-(ru_q+|z|)^2/(2s^2)}\right]\int_u^\infty \phi(x)\,dx \\
&= \frac{1}{2s\sqrt{2\pi}}\left[e^{-(ru_q-|z|)_+^2/(2s^2)}+e^{-(ru_q+|z|)^2/(2s^2)}\right],
\end{aligned}$$

because $\int_u^\infty \phi(x)\,dx = p(u)/2$. Taking the supremum gives

$$\phi(z)\sup_{0<t\le q}\frac{F_{i,z}(t)}{t} \le \frac{1}{2s\sqrt{2\pi}}\left[e^{-(ru_q-|z|)_+^2/(2s^2)}+e^{-(ru_q+|z|)^2/(2s^2)}\right]. \tag{125}$$

Integrating (125) over $|z|\le u_q$ and using symmetry gives

$$\int_{-u_q}^{u_q}\phi(z)\sup_{0<t\le q}\frac{F_{i,z}(t)}{t}\,dz \le \frac{1}{s\sqrt{2\pi}}\left\{\int_0^{u_q}e^{-(ru_q-z)_+^2/(2s^2)}\,dz+\int_0^{u_q}e^{-(ru_q+z)^2/(2s^2)}\,dz\right\}. \tag{126}$$

The two integrals on the right-hand side of (126) are

$$\int_0^{u_q} e^{-(ru_q - z)_+^2/(2s^2)}\,dz = \int_0^{ru_q} e^{-y^2/(2s^2)}\,dy + (1-r)u_q,$$
$$\int_0^{u_q} e^{-(ru_q + z)^2/(2s^2)}\,dz = \int_{ru_q}^{(1+r)u_q} e^{-y^2/(2s^2)}\,dy.$$

Consequently, writing $\lambda = \sqrt{(1-r)/(1+r)} \in (0,1]$,

$$\begin{aligned}\int_{-u_q}^{u_q} \phi(z) \sup_{0<t\le q} \frac{F_{i,z}(t)}{t}\,dz &\le \frac{1}{s\sqrt{2\pi}} \left\{ \int_0^{(1+r)u_q} e^{-y^2/(2s^2)}\,dy + (1-r)u_q \right\} \\ &= \left\{ \Phi\left(\frac{(1+r)u_q}{s}\right) - \frac{1}{2} \right\} + \frac{(1-r)u_q}{s\sqrt{2\pi}} \\ &= \Phi(u_q/\lambda) - \frac{1}{2} + \frac{u_q\lambda}{\sqrt{2\pi}},\end{aligned} \tag{127}$$

where the final equality in (127) uses $s^2 = 1 - r^2 = (1-r)(1+r)$, so $(1+r)/s = 1/\lambda$ and $(1-r)/s = \lambda$. The right-hand side of (127) is increasing in $\lambda \in (0,1]$. Its derivative is

$$\frac{u_q}{\sqrt{2\pi}} \left[1 - \lambda^{-2} \exp\{-u_q^2/(2\lambda^2)\} \right] \ge 0,$$

because $u_q \ge 1$ and, on putting $w = \lambda^{-2} \ge 1$,

$$we^{-u_q^2 w/2} \le we^{-w/2} \le 2/e < 1.$$

Hence (127) is at most

$$\Phi(u_q) - \frac{1}{2} + \frac{u_q}{\sqrt{2\pi}} = \frac{1-q}{2} + \frac{u_q}{\sqrt{2\pi}}. \tag{128}$$

When $r = 1$, the conditional p-value is deterministic. For $|z| < u_q$, it exceeds q, and the boundary $|z| = u_q$ has Lebesgue measure zero; hence (128) remains valid. When $r = 0$, the conditional null p-value is uniform and independent of Z; (128) remains valid, although it is not sharp.

Let $A = \{|Z| \le u_q\}$. Decomposing over A and A^c,

$$\mathrm{FDR} = \mathbb{E}[\mathrm{FDP}\,\mathbf{1}_A] + \mathbb{E}[\mathrm{FDP}\,\mathbf{1}_{A^c}].$$

For the first term, integrate (124) over $z \in [-u_q, u_q]$ and use (128). For the second term, no conditional tail bound is needed: since $0 \le \mathrm{FDP} \le 1$,

$$\mathbb{E}[\mathrm{FDP}\,\mathbf{1}_{A^c}] \le \mathbb{P}(A^c) = \mathbb{P}(|Z| > u_q) = 2\overline{\Phi}(u_q) = q. \tag{129}$$

Combining (124), (128), and (129) yields

$$\mathrm{FDR} \le q + \frac{q}{m} \sum_{i \in \mathcal{H}_0} \left(\frac{1-q}{2} + \frac{u_q}{\sqrt{2\pi}} \right) = q \left\{ 1 + \pi_0 \left(\frac{1-q}{2} + \frac{u_q}{\sqrt{2\pi}} \right) \right\},$$

as claimed. $\square$

4 Discussion

Theorem 2.1 and Proposition 2.2 show that two-sided Gaussian testing under unrestricted correlation can have worse guarantees than the nominal BH level suggests. In particular, the BH procedure can have FDR larger than its nominal level by a factor of order $\sqrt{\log(1/q)}$ as $q \downarrow 0$, so no universal constant multiple of q can bound its FDR over all Gaussian correlation matrices. Because the lower-bound construction belongs to the common-factor class, Theorem 3.1 shows that the worst-case FDR within this class is of exact order $q\sqrt{\log(1/q)}$ as $q \downarrow 0$.

There are, however, other procedures with valid FDR guarantees for dependent two-sided Gaussian tests. One approach is the conditional-calibration framework of Fithian and Lei [2022]. The method decomposes FDR into hypothesis-wise contributions and, for each null i, conditions on a statistic that removes nuisance dependence while leaving a calibrated one-dimensional null distribution for the ith p-value. Their concrete dependence-adjusted BH procedure (dBH) calibrates a hypothesis-specific cutoff for the BH q-value and rejects H_i when its q-value falls below that calibrated cutoff. In general, dBH is at least as powerful as the Benjamini–Yekutieli procedure; under PRDS, it is at least as powerful as BH [Fithian and Lei, 2022]. Thus it provides finite-sample FDR control under known dependence without paying the full harmonic-factor correction of Benjamini and Yekutieli [2001].

Another route is the fixed-X knockoff filter of Barber and Candès [2015]. In a Gaussian linear model, knockoffs augment the design with synthetic variables that mimic the correlation structure of the original variables, and the resulting original-versus-knockoff comparisons create null statistics with the symmetry needed for FDR control. The correlated-Gaussian-testing interpretation is made explicit by Li and Fithian [2021], who recast fixed-X knockoffs as adding user-generated Gaussian noise to a Gaussian estimator to obtain a whitened estimator, followed by conditional inference. Knockoffs can use structural information and flexible feature-importance statistics unavailable to marginal testing. Their weakness is that whitening can require enough added noise to erase much of the signal when the original test statistics are strongly correlated, causing substantial power loss [Li and Fithian, 2021]. Knockoffs also face a threshold phenomenon when the rejection set is small. The calibrated knockoff method of Luo et al. [2025] uses conditional calibration to uniformly improve knockoff procedures, with especially large gains in small-rejection regimes.

More broadly, understanding FDR guarantees under dependence is essential because many real-world test statistics are well approximated by a multivariate Gaussian law. In such settings, the correlation matrix is often structured, but the induced two-sided p-values need not satisfy PRDS or the sign structure required by standard positive-dependence arguments. Two-sided testing can destroy monotonicity even when the underlying Gaussian variables have positive correlations. Procedures that explicitly use dependence information, and theory that clarifies when BH itself is reliable, are therefore both needed.

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A Auxiliary lemmas

This appendix collects the properties of the envelope and Gaussian tails used to define and analyze $\nu_{q,r}$.

The following lemma characterizes the likelihood-ratio envelope.

Lemma A.1. *For $0 \le y \le 1$, $M(y) = 1$. For $y > 1$, $a \mapsto L_a(y)$ has a unique maximizer $a(y) > 0$, characterized by*

$$a(y) = y \tanh\{a(y)y\}. \tag{130}$$

The maps $a : (1, \infty) \to (0, \infty)$ and $M : (1, \infty) \to (1, \infty)$ are continuous and strictly increasing, with

$$\lim_{y \downarrow 1} a(y) = 0, \qquad \lim_{y \to \infty} M(y) = \infty. \tag{131}$$

The inverse $y = y(a)$ is continuously differentiable on $(0, \infty)$, and

$$y'(a) = \frac{1 - y(a)^2 \operatorname{sech}^2\{ay(a)\}}{\tanh\{ay(a)\} + ay(a) \operatorname{sech}^2\{ay(a)\}} > 0. \tag{132}$$

Moreover, with $a_q = a(y_q)$, the deficit in the secondary survival mass satisfies

$$1 - qM\{y(a)\} > 0 \quad (0 < a < a_q), \quad 1 - qM\{y(a_q)\} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}a}[1 - qM\{y(a)\}] < 0. \tag{133}$$

Proof. Let

$$g_y(a) = \log L_a(y) = -\frac{a^2}{2} + \log\cosh(ay).$$

Then

$$g_y'(a) = -a + y\tanh(ay). \tag{134}$$

If $y \le 1$, then $y\tanh(ay) \le ay^2 \le a$, so $M(y) = L_0(y) = 1$.

For $y > 1$, put $t = ay$. Equation $g_y'(a) = 0$ is equivalent to

$$y^2 = \frac{t}{\tanh t}. \tag{135}$$

To verify the strict inequality needed below, let $H(t) = \tanh t - t\operatorname{sech}^2 t$. Then $H(0) = 0$ and $H'(t) = 2t\operatorname{sech}^2 t \tanh t > 0$ for $t > 0$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{\tanh t}\right) = \frac{\tanh t - t\operatorname{sech}^2 t}{\tanh^2 t} > 0.$$

Moreover, $t/\tanh t \to 1$ as $t \downarrow 0$ and $t/\tanh t \to \infty$ as $t \to \infty$. Because $y^2 > 1$, (135) therefore has a unique positive solution. The sign of g_y' changes from positive to negative there, so the solution is the unique maximizer. Moreover,

$$a(y) = \sqrt{t\tanh t}, \quad y = \sqrt{\frac{t}{\tanh t}}, \tag{136}$$

Both parameter functions in (136) are smooth and strictly increasing in $t > 0$. They tend respectively to zero and one as $t \downarrow 0$, and both tend to infinity as $t \to \infty$. Thus $a(y)$ is continuous and strictly increasing on $(1, \infty)$, with $a(y) \to 0$ as $y \downarrow 1$; the inverse function theorem applied to the second parameter function also shows that $a(y)$ is continuously differentiable. Since $M(y) = L_{a(y)}(y)$, the chain rule and $\partial_a L_{a(y)}(y) = 0$ give

$$M'(y) = M(y)a(y)\tanh\{a(y)y\} > 0.$$

Thus M is continuous and strictly increasing. Also

$$M(y) \ge L_y(y) \ge \frac{1}{2}e^{y^2/2} \to \infty.$$

Finally, apply the implicit function theorem to

$$F(a, y) = a - y \tanh(ay) = 0.$$

At the maximizer, with $t = ay > 0$, equation (135) gives

$$y^2 \operatorname{sech}^2(ay) = \frac{t \operatorname{sech}^2 t}{\tanh t} = \frac{2t}{\sinh(2t)} < 1.$$

Consequently,

$$F_a = 1 - y^2 \operatorname{sech}^2(ay) > 0, \qquad F_y = -\tanh(ay) - ay \operatorname{sech}^2(ay) < 0.$$

Thus (132) in Lemma A.1 follows from $y' = -F_a/F_y$.

It remains to prove (133). Since $a_q = a(y_q)$, we have $y(a_q) = y_q$, and therefore $1 - qM\{y(a_q)\} = 0$ by (3). If $0 < a < a_q$, then $y(a) < y_q$; since M is strictly increasing on $(1, \infty)$, $qM\{y(a)\} < qM(y_q) = 1$. Finally,

$$\frac{\mathrm{d}}{\mathrm{d}a}[1 - qM\{y(a)\}] = -qM'\{y(a)\}y'(a) < 0,$$

because $M' > 0$ on $(1, \infty)$ and $y' > 0$ on $(0, \infty)$. $\square$

The final lemma collects the Mills bounds and quantile asymptotics used to control Gaussian tails.

Lemma A.2. *The normal hazard $\lambda(x) = \phi(x)/\overline{\Phi}(x)$ is increasing on $\mathbb{R}$ and satisfies*

$$x < \lambda(x) < x + x^{-1} \quad (x > 0). \tag{137}$$

Moreover, as $x \to \infty$,

$$\lambda(x) = x + x^{-1} + O(x^{-3}). \tag{138}$$

As $x \to \infty$,

$$\overline{\Phi}(x) = \frac{\phi(x)}{x}\{1 + o(1)\}. \tag{139}$$

For every fixed $0 < \sigma < 1$, as $x \to \infty$,

$$\frac{\overline{\Phi}(\sigma x)}{\overline{\Phi}(x)} \sim \frac{1}{\sigma} \exp\left\{\frac{(1 - \sigma^2)x^2}{2}\right\}. \tag{140}$$

For every $0 < H < \infty$, uniformly over $|u| \vee |v| \le H$, as $x \to \infty$,

$$\frac{p(x + u)}{p(x + v)} = \frac{x + v}{x + u} \exp\left\{-x(u - v) - \frac{u^2 - v^2}{2}\right\}\{1 + o(1)\}. \tag{141}$$

If $c > 0$ is fixed, then, as $q \downarrow 0$,

$$p^{-1}(cq) \sim \sqrt{2 \log(1/q)}. \tag{142}$$

In particular, for every fixed $c_1, c_2 > 0$,

$$\frac{p^{-1}(c_1 q)}{p^{-1}(c_2 q)} \longrightarrow 1 \quad (q \downarrow 0).$$

Finally, for $x \ge 0$,

$$2\overline{\Phi}(x) \le e^{-x^2/2}. \tag{143}$$

For $x > 0$ and $\delta \ge 0$,

$$\frac{p(x + \delta)}{p(x)} \le \left(1 + x^{-2}\right) \exp\left\{-x\delta - \frac{\delta^2}{2}\right\}. \tag{144}$$

Consequently, if $x \to \infty$, then the following tail-ratio limits hold.

(i) *For every* $0 < H < \infty$*, uniformly over* $|u| \vee |v| \leq H$*,*

$$\log \frac{p(x+u)}{p(x+v)} = -x(u-v) - \frac{u^2 - v^2}{2} + o(1). \tag{145}$$

(ii) *For every* $0 < H < \infty$*, uniformly over* $|h| \leq H$*,*

$$\frac{p(x)}{p(x+h/x)} \longrightarrow e^h. \tag{146}$$

(iii) *If* $0 < \delta_0 \leq \delta \leq \delta_1 < \infty$*, then*

$$\frac{p(x+\delta)}{p(x)} \longrightarrow 0 \tag{147}$$

uniformly in δ*.*

(iv) *If* $\delta_x \geq 0$ *and* $x\delta_x \to \infty$*, then*

$$\frac{p(x+\delta_x)}{p(x)} \longrightarrow 0. \tag{148}$$

Proof. For $x > 0$, the upper Mills bound follows from

$$\overline{\Phi}(x) < \frac{1}{x}\int_x^\infty t\phi(t)\ \mathrm{d}t = \frac{\phi(x)}{x}. \tag{149}$$

Integration by parts gives

$$\overline{\Phi}(x) = \frac{\phi(x)}{x} - \int_x^\infty \frac{\phi(t)}{t^2}\ \mathrm{d}t > \frac{\phi(x)}{x} - \frac{\overline{\Phi}(x)}{x^2},$$

which proves the lower Mills bound

$$\frac{x}{1+x^2}\phi(x) < \overline{\Phi}(x). \tag{150}$$

A further integration by parts gives

$$\overline{\Phi}(x) = \frac{\phi(x)}{x} - \frac{\phi(x)}{x^3} + 3\int_x^\infty \frac{\phi(t)}{t^4}\ \mathrm{d}t,$$

where

$$0 < \int_x^\infty \frac{\phi(t)}{t^4}\ \mathrm{d}t < \frac{\phi(x)}{x^5}.$$

Hence

$$\overline{\Phi}(x) = \frac{\phi(x)}{x}\{1 - x^{-2} + O(x^{-4})\},$$

and inversion proves (138) in Lemma A.2. Equations (149) and (150) imply (137) in Lemma A.2 for $x > 0$. Since $\lambda(x) - x > 0$ also holds trivially for $x \leq 0$, the identity

$$\lambda'(x) = \lambda(x)\{\lambda(x) - x\}$$

shows that λ is increasing on $\mathbb{R}$.

Equations (149) and (150) give (139) in Lemma A.2. Applying this at x and σx proves (140) in Lemma A.2. Applying (139) in Lemma A.2 at $x+u$ and $x+v$ gives (141) in Lemma A.2, for every $0 < H < \infty$, uniformly over $|u| \vee |v| \leq H$. Moreover, as $x \to \infty$,

$$-\log p(x) = \frac{x^2}{2} + O(\log x),$$

which yields (142) in Lemma A.2 by inversion. To prove $2\overline{\Phi}(x) \le e^{-x^2/2}$, set

$$d(x) = e^{-x^2/2} - 2\overline{\Phi}(x).$$

Then $d(0) = 0$, $\lim_{x\to\infty} d(x) = 0$, and

$$d'(x) = e^{-x^2/2}\left(\sqrt{\frac{2}{\pi}} - x\right).$$

Thus d first increases and then decreases to zero, so $d(x) \ge 0$ for all $x \ge 0$. Finally, (149) and (150) imply

$$\frac{p(x+\delta)}{p(x)} \le \frac{1+x^2}{x(x+\delta)} \exp\left\{-x\delta - \frac{\delta^2}{2}\right\} \le \left(1 + x^{-2}\right) \exp\left\{-x\delta - \frac{\delta^2}{2}\right\},$$

which proves (144) in Lemma A.2.

It remains to prove the four stated tail-ratio consequences. Taking logarithms in (141) in Lemma A.2 gives (145) in Lemma A.2, because

$$\log\frac{x+v}{x+u} = o(1)$$

for every $0 < H < \infty$, uniformly over $|u| \vee |v| \le H$. For (146) in Lemma A.2, apply (141) in Lemma A.2 with $u = 0$ and $v = h/x$. For every $0 < H < \infty$, uniformly over $|h| \le H$,

$$\frac{p(x)}{p(x+h/x)} = \frac{x+h/x}{x} \exp\left\{h + \frac{h^2}{2x^2}\right\}\{1+o(1)\} \longrightarrow e^h.$$

For (147) in Lemma A.2, apply (145) in Lemma A.2 with $u = \delta$ and $v = 0$. If $0 < \delta_0 \le \delta \le \delta_1 < \infty$, then

$$\log\frac{p(x+\delta)}{p(x)} \le -x\delta_0 + O(1) \longrightarrow -\infty$$

uniformly in δ. Finally, (148) in Lemma A.2 follows from (144) in Lemma A.2 with $\delta = \delta_x$, because $x\delta_x \to \infty$. $\square$