How Much Can Gaussian Dependence Inflate the Benjamini–Hochberg Procedure's FDR?

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Abstract

We study the worst-case false discovery rate (FDR) of the Benjamini–Hochberg procedure for both one- and two-sided Gaussian tests when the correlation matrix is otherwise unrestricted. In each setting we construct a q-indexed family of finite Gaussian models whose FDR divided by q diverges as $q \downarrow 0$, disproving any universal multiplicative FDR bound. For two-sided tests, the supremum over the number of hypotheses, mean vector, and correlation matrix is at least an explicit $\ell_{=}(q) > q$ satisfying

$$\ell_{=}(q) = \frac{q\sqrt{\log(1/q)}}{2\sqrt{\pi}} + c_\ell q + o(q), \qquad c_\ell = 0.6492828\ldots.$$

For the one-sided hypotheses $H_i : \theta_i \le 0$, a sign-reversed one-common-factor construction gives the stronger explicit lower bound $\ell_{\le}(q) > q$, with

$$\ell_{\le}(q) = \frac{q\sqrt{\log(1/q)}}{\sqrt{\pi}} + \frac{q}{2} + o(q).$$

Finally, we prove sharp leading upper bounds for the one-common-factor class. They match the two-sided constant $1/(2\sqrt{\pi})$ and the one-sided constant $1/\sqrt{\pi}$ in the $q\sqrt{\log(1/q)}$ normalization.

1 Introduction

We observe a Gaussian vector $T = (T_1, \ldots, T_N)$ with mean θ and covariance matrix Σ, where $\Sigma_{ii} = 1$ for $i = 1, \ldots, N$. We consider two testing problems. For two-sided tests, $H_i : \theta_i = 0$ and

$$P_i = p(|T_i|), \qquad p(x) = 2\overline{\Phi}(x) = 2\{1 - \Phi(x)\}.$$

For one-sided tests, $H_i : \theta_i \le 0$ and

$$P_i = p_+(T_i), \qquad p_+(x) = \overline{\Phi}(x) = 1 - \Phi(x).$$

Here Φ and $\phi = \Phi'$ are the standard normal CDF and density. The normal hazard is

$$\lambda(x) = \frac{\phi(x)}{\overline{\Phi}(x)}. \tag{1}$$

We use the conventions $p^{-1}(0) = p_+^{-1}(0) = \infty$. The Benjamini–Hochberg procedure [Benjamini and Hochberg, 1995] at level q orders the p-values as

$$P_{(1)} \le \cdots \le P_{(N)}$$

and rejects the hypotheses corresponding to

$$P_{(1)}, \ldots, P_{(R)}, \quad R = \max\{1 \le k \le N : P_{(k)} \le qk/N\},$$

with $R = 0$ if the set is empty. If V is the number of rejected true nulls, then the false discovery proportion (FDP) and false discovery rate (FDR) are

$$\mathrm{FDP} = \frac{V}{R \vee 1}, \quad \mathrm{FDR} = \mathbb{E}(\mathrm{FDP}).$$

A long-standing conjecture for correlated two-sided Gaussian z-tests is that BH controls the FDR at its nominal level q, regardless of the correlation matrix [Reiner-Benaim, 2007, Benjamini, 2010, Roux, 2018, Sarkar, 2023, Sarkar and Zhang, 2025]. Benjamini [2010, p. 407] summarized the evidence as follows:

> "The modification to general dependence is often not needed: convincing simutheoretical evidence indicates that the same holds for two-sided z-tests with any correlation structure."

Benjamini then noted that a complete theoretical proof was still missing. These works stated or supported this conjecture. However, recently, Dobriban [2026] found a counterexample where $\mathrm{FDR} > 0.0104$ when $q = 0.01$.

In this paper, we ask a more refined question in each testing problem: how large can the FDR be as a function of q? For two-sided tests, we seek a lower bound $\ell_{=}(q)$ such that

$$\sup_{\substack{N\in\mathbb{N},\ \theta\in\mathbb{R}^N,\ \Sigma\in\mathbb{S}_+^N \\ \Sigma_{ii}=1,\ 1\le i\le N}} \mathrm{FDR}_{\theta,\Sigma}(\mathrm{BH}_q) \ge \ell_{=}(q). \tag{2}$$

Here $\mathbb{S}_+^N$ denotes the cone of $N \times N$ symmetric positive semidefinite matrices. A relaxed folklore conjecture is that the worst-case FDR in (2) is bounded above by Cq for some universal constant $C > 0$. However, Theorem 2.1 and Proposition 2.2 together show that the worst-case FDR is at least a quantity satisfying, as $q \downarrow 0$,

$$\ell_{=}(q) = \frac{q\sqrt{\log(1/q)}}{2\sqrt{\pi}} + c_\ell q + o(q), \qquad c_\ell = 0.6492828\ldots.$$

In particular, the ratio of the supremum in (2) to q diverges as $q \downarrow 0$.

Moreover, we derive an exact formula for $\ell_{=}(q)$. We prove that $\ell_{=}(q) > q$ for every $q \in (0,1)$. Instead of seeking a counterexample for a given nominal level q, Theorem 2.1 and Proposition 2.3 together disprove the conjecture that BH controls the FDR for correlated two-sided Gaussian tests. Figure 1 illustrates the resulting two-sided FDR inflation.

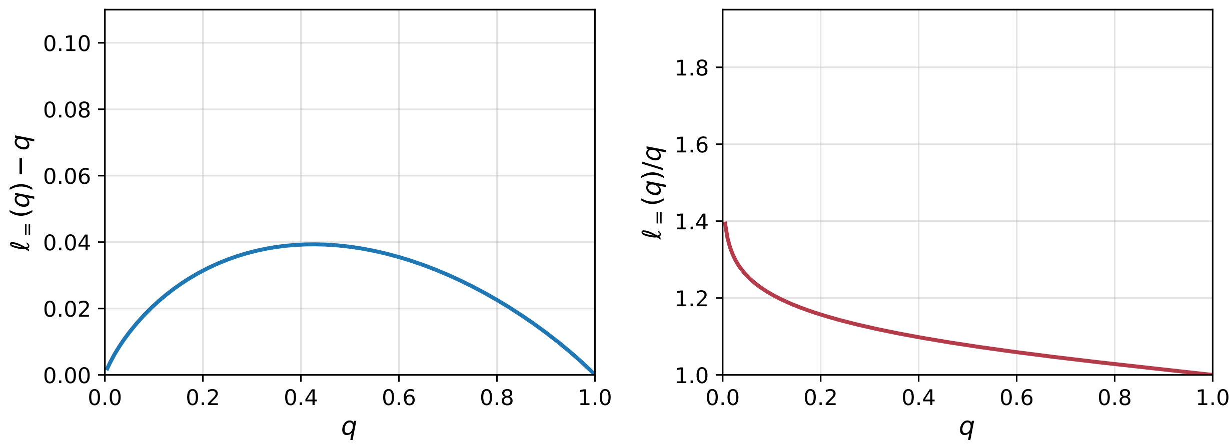

Figure 1: Numerical evaluation of $\ell_=(q) - q$ and $\ell_=(q)/q$ over the grid $q = 0.005k$, $k = 1, \ldots, 199$.

q	0.01	0.05	0.10	0.20
$\ell_=(q) - q$	0.0036	0.0129	0.0209	0.0314
$\ell_=(q)/q$	1.3553	1.2571	1.2094	1.1570

Table 1: The absolute inflation $\ell_=(q) - q$ and relative inflation $\ell_=(q)/q$ at commonly used nominal levels.

For one-sided Gaussian tests, the classical negatively correlated examples of Hochberg and Rom [1995] and Samuel-Cahn [1996] show that the Simes inequality, and hence BH under the global null, can be anticonservative. Chi et al. [2025] prove useful upper bounds under several notions of negative dependence, but the adversarial construction used in their sharpness comparison, due to Su [2018], is not jointly Gaussian. Our one-sided construction instead retains positively correlated nulls and reverses the loading of the non-nulls. Theorem 3.1 and Proposition 3.3 give, for every $q \in (0, 1)$,

$$\ell_{\le}(q) = \frac{q\sqrt{\log(1/q)}}{\sqrt{\pi}} + \frac{q}{2} + \overline{\Phi}\left(\sqrt{2\log(1/q)}\right) > q,$$

and Proposition 3.2 gives

$$\ell_{\le}(q) = \frac{q\sqrt{\log(1/q)}}{\sqrt{\pi}} + \frac{q}{2} + o(q).$$

This is a larger leading constant than in the two-sided construction and again implies that the worst-case FDR ratio diverges. Figure 2 illustrates the resulting one-sided FDR inflation.

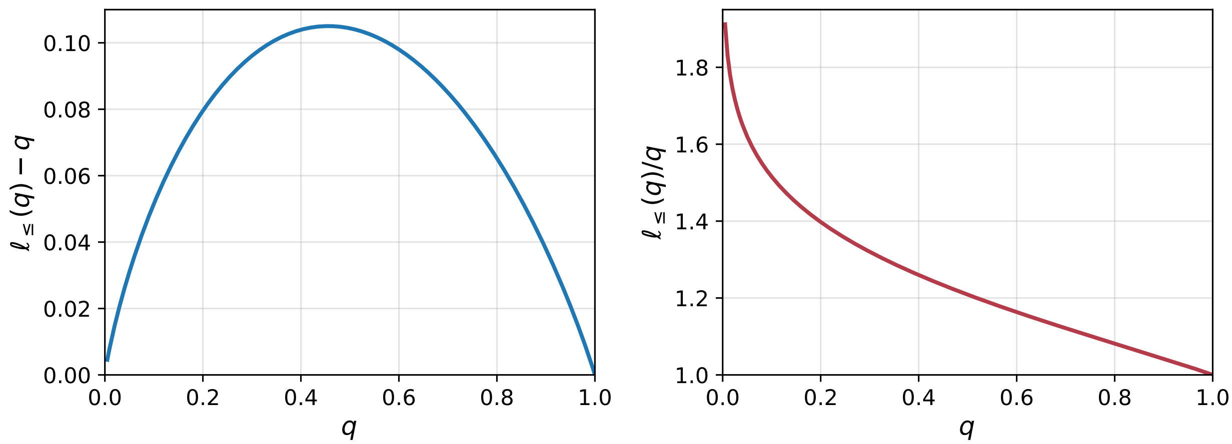

Figure 2: Numerical evaluation of $\ell_{\leq}(q) - q$ and $\ell_{\leq}(q)/q$ over the grid $q = 0.005k$, $k = 1, \ldots, 199$.

Table 2: The absolute inflation $\ell_{\leq}(q) - q$ and relative inflation $\ell_{\leq}(q)/q$ at commonly used nominal levels.

q	0.01	0.05	0.10	0.20
$\ell_{\leq}(q) - q$	0.00831	0.03101	0.05155	0.07955
$\ell_{\leq}(q)/q$	1.8311	1.6203	1.5155	1.3977

The dependence mechanisms differ sharply between the two constructions. Every covariance in the two-sided construction is positive, but the map $T_i \mapsto 2\overline{\Phi}(|T_i|)$ is not monotone and need not make the full vector of two-sided p-values satisfy positive regression dependence on a subset (PRDS) with respect to the true nulls. For one-sided Gaussian p-values, by contrast, nonnegative correlations between every true null and every coordinate imply that the full p-value vector is PRDS on the subset of true nulls and hence FDR control [Benjamini and Yekutieli, 2001]. Negative dependence touching a true null is therefore essential in the one-sided construction: null–null correlations are positive, while null–non-null correlations are negative. Thus, the full vector of one-sided p-values in our construction is not PRDS; Section 3 verifies this directly.

The lower-bound rate $q\sqrt{\log(1/q)}$ should be compared with the FDR-linking theorem of Su [2018]. Positive regression dependence within nulls (PRDN) relaxes PRDS by imposing the relevant positive dependence assumption only on the null p-values. Under PRDN, the FDR-linking argument gives an $O(q\log(1/q))$ bound. Section 4 proves sharper Gaussian bounds. For two-sided one-common-factor models,

$$\sup_{\substack{m\in\mathbb{N},\ \theta\in\mathbb{R}^m,\\ a\in[-1,1]^m}} \mathrm{FDR}_{\theta,a}(\mathrm{BH}_q) = \left\{\frac{1}{2\sqrt{\pi}} + o(1)\right\} q\sqrt{\log(1/q)},$$

so the lower construction attains the sharp leading constant in this class. This bound improves on the general PRDN linking rate. Moreover, as explained in Remark 4.9, the two-sided null p-values in this class satisfy the PRDN property by the absolute-value Gaussian multivariate total positivity of order two (MTP_2) criterion of Karlin and Rinott [1981]. For one-sided one-common-factor models, Theorem 4.10 and Corollary 4.11 give

$$\sup_{\substack{m\in\mathbb{N},\ \theta\in\mathbb{R}^m,\\ a\in[-1,1]^m}} \mathrm{FDR}_{\theta,a}(\mathrm{BH}_q) = \left\{\frac{1}{\sqrt{\pi}} + o(1)\right\} q\sqrt{\log(1/q)}.$$

This matches the leading constant in $\ell_{\leq}(q)$. Both bounds are uniform over the number of hypotheses, means, and scalar loadings.

We conclude the section with a new conjecture that neither the author nor GPT 5.6 Sol was able to disprove after significant efforts.

Conjecture 1. *The unrestricted Gaussian worst-case FDR satisfies*

$$\lim_{q\downarrow 0} \frac{\sup_{\substack{N\in\mathbb{N},\ \theta\in\mathbb{R}^N,\ \Sigma\in\mathbb{S}_+^N\\ \Sigma_{ii}=1,\ 1\leq i\leq N}} \mathrm{FDR}_{\theta,\Sigma}(\mathrm{BH}_q)}{q\sqrt{\log(1/q)}} = \begin{cases} \dfrac{1}{2\sqrt{\pi}}, & \textit{for two-sided tests,}\\ \dfrac{1}{\sqrt{\pi}}, & \textit{for one-sided tests.}\end{cases}$$

2 Lower bound for two-sided tests

2.1 Main result

$$M(y) = \sup_{a\geq 0} e^{-a^2/2}\cosh(ay). \tag{3}$$

Appendix A identifies the likelihood-ratio envelope: $M = 1$ on $[0,1]$, and M is continuous, strictly increasing, and unbounded on $(1,\infty)$. Hence there is a unique $y_q > 1$ such that

$$qM(y_q) = 1. \tag{4}$$

Set

$$\ell_{=}(q) = q\Phi(1) + q\int_1^{y_q} M(y)\phi(y)\,\mathrm{d}y + \overline{\Phi}(y_q). \tag{5}$$

The next theorem shows that the lower bound can be approached arbitrarily closely by finite-dimensional Gaussian models with positive definite correlation matrices.

Theorem 2.1. *For $T \sim N(\theta,\Sigma)$, let $\mathrm{FDR}_{\theta,\Sigma}(\mathrm{BH}_q)$ denote the FDR of BH_q applied to the two-sided p-values $p(|T_1|),\ldots,p(|T_N|)$. For every $q\in(0,1)$ and $\delta>0$, there are $N\in\mathbb{N}$, $\theta\in\mathbb{R}^N$, and a positive definite correlation matrix Σ such that*

$$\mathrm{FDR}_{\theta,\Sigma}(\mathrm{BH}_q) > \ell_{=}(q) - \delta. \tag{6}$$

Consequently,

$$\sup_{\substack{N\in\mathbb{N},\ \theta\in\mathbb{R}^N,\\ \Sigma\succ 0,\ \mathrm{diag}(\Sigma)=\mathbf{1}_N}} \mathrm{FDR}_{\theta,\Sigma}(\mathrm{BH}_q) \geq \ell_{=}(q). \tag{7}$$

The next proposition gives the small-q expansion of the lower bound.

Proposition 2.2. *As $q\downarrow 0$,*

$$\ell_{=}(q) = \frac{q\sqrt{\log(1/q)}}{2\sqrt{\pi}} + c_\ell q + o(q),$$

where

$$\begin{aligned} c_\ell &= \Phi(1) - \frac{1}{2\sqrt{2\pi}} + \int_1^\infty \left\{M(y)\phi(y) - \frac{1}{2\sqrt{2\pi}}\right\}\mathrm{d}y \\ &= 0.6492828\ldots. \end{aligned}$$

Proposition 2.3 further shows that $\ell_=(q) > q$ for every $q \in (0,1)$. Together with Theorem 2.1, this implies that BH fails to control the FDR at every nontrivial nominal level in the correlated two-sided Gaussian setting.

Proposition 2.3. *For every $q \in (0,1)$,*

$$\ell_=(q) > q.$$

Figure 1 in the Introduction displays the numerical behavior of the two-sided lower bound. Table 1 gives the absolute inflation $\ell_=(q) - q$ and relative inflation $\ell_=(q)/q$ for commonly used nominal levels.

2.2 Roadmap

Fix $q \in (0,1)$ and a design law ν_q. Let $\Omega_N = ((\Theta_1, B_1), \ldots, (\Theta_N, B_N))$, where the pairs are i.i.d. draws from ν_q, independently of $Z, \varepsilon_1, \ldots, \varepsilon_N \overset{\text{i.i.d.}}{\sim} N(0,1)$, and set

$$T_i = \Theta_i + B_i Z + \sqrt{1 - B_i^2}\,\varepsilon_i, \quad i = 1, \ldots, N.$$

Conditional on a realization of Ω_N, this is a finite deterministic Gaussian design. Since BH is a step-up procedure, its rejection set is $\{i : |T_i| \geq \widehat{C}_N\}$. We will show in Subsection 2.4 that, conditional on $Z = z$, $\widehat{C}_N \to C(z)$ and FDP $\to D(z)$. Consequently,

$$\mathbb{E}_{\Omega_N}[\mathrm{FDR}_{\Omega_N}(\mathrm{BH}_q)] = \mathbb{E}_{\Omega_N, Z, \varepsilon}[\mathrm{FDP}] \longrightarrow \mathbb{E}[D(Z)].$$

Thus, for any $\delta > 0$ and all sufficiently large N, at least one deterministic realization ω_N of Ω_N satisfies

$$\mathrm{FDR}_{\omega_N}(\mathrm{BH}_q) \geq \mathbb{E}[D(Z)] - \delta.$$

We next construct a family of laws $\nu_{q,r}$. Write D_r for the conditional FDP limit associated with $\nu_{q,r}$. For every $z \notin \{-1, -y_q\}$, we prove that $D_r(z) \to D_*(z)$ as $r \downarrow 0$, where

$$D_*(z) = \begin{cases} q, & z \geq -1, \\ qM(-z), & -y_q < z < -1, \\ 1, & z \leq -y_q. \end{cases}$$

Because $0 \leq D_r, D_* \leq 1$, dominated convergence then gives

$$\mathbb{E}[D_r(Z)] \longrightarrow \mathbb{E}[D_*(Z)] = \ell_=(q).$$

The argument therefore takes the iterated limit $\lim_{r\downarrow 0} \lim_{N\to\infty}$: first $N \to \infty$ for each fixed r, and then $r \downarrow 0$. To obtain a single finite example, we choose r sufficiently small and then N sufficiently large. A small independent Gaussian perturbation finally makes its covariance matrix positive definite.

2.3 Population quantities

Put $u_q = p^{-1}(q)$. Let ν_q be a probability measure on $\mathbb{R} \times [-1,1]$. Write

$$\pi_0 = \nu_q(\{0\} \times [-1,1]), \tag{8}$$

and assume $\pi_0 > 0$.

For $(\theta, b) \in \mathbb{R} \times [-1, 1]$, $z \in \mathbb{R}$, and $s \in (0, 1]$, let $u_s = p^{-1}(s)$ and define

$$F_{\theta,b,z}(s) = \mathbb{P}\{p(|T_i|) \le s \,|\, \Theta_i = \theta,\ B_i = b,\ Z = z\}\,.$$

If $|b| < 1$, then

$$F_{\theta,b,z}(s) = \overline{\Phi}\left(\frac{u_s - \theta - bz}{\sqrt{1-b^2}}\right) + \overline{\Phi}\left(\frac{u_s + \theta + bz}{\sqrt{1-b^2}}\right). \tag{9}$$

If $|b| = 1$, then

$$F_{\theta,b,z}(s) = \mathbf{1}\{p(|\theta + bz|) \le s\}. \tag{10}$$

Set $F_{\theta,b,z}(0) = 0$.

Conditional on $Z = z$, the CDFs of the mixture p-value and the null p-value are

$$R_z(s) = \int F_{\theta,b,z}(s)\, \nu_q(\,\mathrm{d}\theta,\, \mathrm{d}b), \tag{11}$$

$$R_{0,z}(s) = \frac{1}{\pi_0} \int_{\{0\}\times[-1,1]} F_{0,b,z}(s)\, \nu_q(\,\mathrm{d}\theta,\, \mathrm{d}b). \tag{12}$$

For $c \ge u_q$, define the normalized quantities

$$\bar{R}_z(c) = \frac{qR_z\{p(c)\}}{p(c)}, \tag{13}$$

$$\bar{R}_{0,z}(c) = \frac{q\pi_0 R_{0,z}\{p(c)\}}{p(c)}. \tag{14}$$

Then the limiting cutoff can be written as

$$C(z) = \inf\{c \ge u_q : \bar{R}_z(c) \ge 1\}. \tag{15}$$

Here and below, $\inf \varnothing = \infty$. The limiting conditional FDP is defined by

$$D(z) = \begin{cases} \bar{R}_{0,z}\{C(z)\}, & C(z) < \infty, \\ 0, & C(z) = \infty. \end{cases} \tag{16}$$

We call a finite cutoff a regular first crossing when it satisfies the following requirements.

Definition 2.4. A finite cutoff $C(z) > u_q$ is regular if

$$\sup_{u_q \le c \le C(z)-\rho} \bar{R}_z(c) < 1 \qquad \text{for every } 0 < \rho < C(z) - u_q, \tag{17}$$

$$\sup_{C(z) < c < C(z)+\rho} \bar{R}_z(c) > 1 \qquad \text{for every } \rho > 0, \tag{18}$$

and $c \mapsto \bar{R}_{0,z}(c)$ is continuous at $C(z)$.

The following proposition reduces regularity to a local monotonicity check.

Proposition 2.5. *Let $C(z) > u_q$ be finite. Suppose there exist $u_q < a < C(z) < b$ such that the map $c \mapsto \bar{R}_z(c)$ is differentiable on (a, b) and satisfies*

$$\bar{R}_z'(c) > 0, \qquad a < c < b. \tag{19}$$

Suppose also that $c \mapsto \bar{R}_{0,z}(c)$ is continuous at $C(z)$. Then $C(z)$ is regular.

Proof. The map $c \mapsto R_z\{p(c)\}$ is a survival function and hence is upper semicontinuous. Since p is positive and continuous, $c \mapsto \bar{R}_z(c)$ is also upper semicontinuous. By the definition of $C(z)$, $\bar{R}_z(c) < 1$ for every $c < C(z)$. Therefore, for every $0 < \rho < C(z) - u_q$, upper semicontinuity and compactness give

$$\sup_{u_q \le c \le C(z)-\rho} \bar{R}_z(c) < 1,$$

which proves (17).

By the definition of $C(z)$, there is a sequence $c_n \ge C(z)$ decreasing to $C(z)$ such that $\bar{R}_z(c_n) \ge 1$. Continuity on (a,b) gives $\bar{R}_z\{C(z)\} \ge 1$. On the other hand, for any sequence $d_n < C(z)$ increasing to $C(z)$, the first-crossing definition gives $\bar{R}_z(d_n) < 1$, so continuity gives $\bar{R}_z\{C(z)\} \le 1$. Hence $\bar{R}_z\{C(z)\} = 1$. The mean value theorem and (19) imply strict monotonicity on (a,b), which yields

$$\bar{R}_z(c) > 1 \quad \text{for } C(z) < c < b.$$

This gives (18). Continuity of the null contribution at $C(z)$ completes Definition 2.4. $\square$

2.4 Limiting cutoff and FDR

Let

$$\Omega_N = ((\Theta_1, B_1), \dots, (\Theta_N, B_N)), \quad (\Theta_i, B_i) \overset{\text{i.i.d.}}{\sim} \nu_q. \tag{20}$$

Since the distribution of $(T_1, \dots, T_N)$ is fully determined by Ω_N, we write $\mathrm{FDR}_{\theta,\Sigma}(\mathrm{BH}_q)$ as $\mathrm{FDR}_{\Omega_N}(\mathrm{BH}_q)$.

Define $P_i = p(|T_i|)$.

$$\widehat{R}_N(s) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\{P_i \le s\}, \tag{21}$$

$$\widehat{R}_{0,N}(s) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}\{\Theta_i = 0,\ P_i \le s\}. \tag{22}$$

The following lemma gives a uniform conditional empirical-CDF bound. For a function f on $[0,1]$, write $\|f\|_{\infty,[0,1]} = \sup_{0\le s\le 1} |f(s)|$.

Lemma 2.6. *For every $z \in \mathbb{R}$ and $t > 0$,*

$$\mathbb{P}\Big(\max\Big\{\|\widehat{R}_N - R_z\|_{\infty,[0,1]}, \|\widehat{R}_{0,N} - \pi_0 R_{0,z}\|_{\infty,[0,1]}\Big\} > t \,\Big|\, Z = z\Big) \le 4e^{-2Nt^2}. \tag{23}$$

Proof. Conditional on $Z = z$, the P_i are i.i.d. with CDF R_z. Also define

$$W_i = P_i \mathbf{1}\{\Theta_i = 0\} + 2\mathbf{1}\{\Theta_i \ne 0\}.$$

For $0 \le s \le 1$,

$$\mathbf{1}\{W_i \le s\} = \mathbf{1}\{\Theta_i = 0,\ P_i \le s\}, \quad \mathbb{P}(W_i \le s \mid Z = z) = \pi_0 R_{0,z}(s).$$

Apply the Dvoretzky–Kiefer–Wolfowitz inequality twice and take a union bound. $\square$

The next lemma gives an exact characterization of the BH cutoff and FDP through the empirical CDFs.

Lemma 2.7. *Let R_N be the number of BH rejections and put*

$$\widehat{\tau}_N = \frac{qR_N}{N}. \tag{24}$$

If $R_N \geq 1$, then

$$\widehat{R}_N(\widehat{\tau}_N) = \frac{R_N}{N}, \quad \mathrm{FDP} = \frac{q\widehat{R}_{0,N}(\widehat{\tau}_N)}{\widehat{\tau}_N}. \tag{25}$$

With $\widehat{C}_N = p^{-1}(\widehat{\tau}_N)$,

$$\widehat{C}_N = \inf\left\{c \geq u_q : \frac{q\widehat{R}_N\{p(c)\}}{p(c)} \geq 1\right\}. \tag{26}$$

Then the BH rejection set is $\{i : |T_i| \geq \widehat{C}_N\}$. If $R_N = 0$, the set in (26) is empty and $\widehat{C}_N = \infty$.

Proof. Let $P_{(1)} \leq \cdots \leq P_{(N)}$. If more than R_N values were at most qR_N/N, then

$$P_{(R_N+1)} \leq qR_N/N < q(R_N+1)/N,$$

contradicting the maximality of R_N. Hence $N\widehat{R}_N(\widehat{\tau}_N) = R_N$, and (25) follows. Since p is decreasing, the BH step-up definition is equivalent to (26). □

The next theorem establishes the limiting cutoff conditional on the common factor Z.

Theorem 2.8. *Assume that $C(z)$ is regular for almost every z. Then, for every z at which $C(z)$ is regular, conditional on $Z = z$,*

$$\widehat{C}_N \longrightarrow C(z), \quad \mathrm{FDP} \longrightarrow D(z) = \bar{R}_{0,z}\{C(z)\} = \frac{q\pi_0 R_{0,z}[p\{C(z)\}]}{p\{C(z)\}} \tag{27}$$

in probability. Consequently,

$$\lim_{N\to\infty} \mathbb{E}_{\Omega_N}[\mathrm{FDR}_{\Omega_N}(\mathrm{BH}_q)] = \mathbb{E}[D(Z)], \tag{28}$$

where

$$\mathrm{FDR}_{\Omega_N}(\mathrm{BH}_q) = \mathbb{E}_{Z,\varepsilon}[\mathrm{FDP} \mid \Omega_N]. \tag{29}$$

Proof. Fix a regular z and $\rho, \tau > 0$. Since $D(z) = \bar{R}_{0,z}\{C(z)\}$ and $c \mapsto \bar{R}_{0,z}(c)$ is continuous at $C(z)$, choose

$$0 < \rho(z) < \min\{\rho, C(z) - u_q\}$$

so that

$$\sup_{C(z)-\rho(z)\leq c\leq C(z)+\rho(z)} |\bar{R}_{0,z}(c) - D(z)| \leq \tau. \tag{30}$$

Set $A(z) = C(z) - \rho(z)$. Equation (17) gives $\sup_{u_q\leq c\leq A(z)} \bar{R}_z(c) < 1$, and (18) permits a choice of $B(z) \in (C(z), C(z)+\rho(z))$ with $\bar{R}_z\{B(z)\} > 1$. Set

$$\gamma(z) = \frac{1}{2}\min\left\{1 - \sup_{u_q\leq c\leq A(z)} \bar{R}_z(c), \quad \bar{R}_z\{B(z)\} - 1\right\} > 0.$$

Put

$$E_N(z) = \max\left\{\|\widehat{R}_N - R_z\|_{\infty,[0,1]}, \|\widehat{R}_{0,N} - \pi_0 R_{0,z}\|_{\infty,[0,1]}\right\}.$$

Lemma 2.6 gives $E_N(z) \to 0$ in probability conditional on $Z = z$. For every $u_q \le c \le B(z)$,

$$\begin{aligned} \left| \frac{q\widehat{R}_N\{p(c)\}}{p(c)} - \bar{R}_z(c) \right| &\le \frac{qE_N(z)}{p\{B(z)\}}, \\ \left| \frac{q\widehat{R}_{0,N}\{p(c)\}}{p(c)} - \bar{R}_{0,z}(c) \right| &\le \frac{qE_N(z)}{p\{B(z)\}}. \end{aligned} \tag{31}$$

By the empirical curve we mean

$$c \longmapsto \frac{q\widehat{R}_N\{p(c)\}}{p(c)}.$$

On the event $qE_N(z)/p\{B(z)\} < \gamma(z)$, the definition of $\gamma(z)$ and (31) give

$$\begin{aligned} \sup_{u_q \le c \le A(z)} \frac{q\widehat{R}_N\{p(c)\}}{p(c)} &\le \sup_{u_q \le c \le A(z)} \bar{R}_z(c) + \frac{qE_N(z)}{p\{B(z)\}} \\ &< 1 - 2\gamma(z) + \gamma(z) < 1, \end{aligned}$$

whereas

$$\frac{q\widehat{R}_N\{p(B(z))\}}{p(B(z))} \ge \bar{R}_z\{B(z)\} - \frac{qE_N(z)}{p\{B(z)\}} > 1 + 2\gamma(z) - \gamma(z) > 1.$$

Lemma 2.7 and (30) therefore give

$$A(z) < \widehat{C}_N \le B(z), \qquad |\,\mathrm{FDP} - D(z)| \le \tau + \frac{qE_N(z)}{p\{B(z)\}}.$$

Because $\rho, \tau > 0$ were arbitrary, this proves (27) at every regular z.

For completeness, the quantities in the expectation can be taken to be measurable without making a measurable selection of $B(z)$. The maps $(z, c) \mapsto \bar{R}_z(c)$ and $(z, c) \mapsto \bar{R}_{0,z}(c)$ are jointly Borel measurable, and the first is left-continuous in c. Define

$$C^\circ(z) = \inf\{c \in \mathbb{Q} : c \ge u_q,\ \bar{R}_z(c) > 1\}.$$

At every regular z, left continuity and (18) imply $C^\circ(z) = C(z)$. Thus C has a measurable version. Moreover, for $c < C(z)$, $\bar{R}_{0,z}(c) \le \bar{R}_z(c) < 1$; continuity of the null contribution at the crossing gives $0 \le D(z) \le 1$. Hence

$$D^\circ(z) = \begin{cases} \min[1, \bar{R}_{0,z}\{C^\circ(z)\}], & C^\circ(z) < \infty, \\ 0, & C^\circ(z) = \infty, \end{cases}$$

is a measurable $[0, 1]$-valued version of $D(z)$ at almost every z. Use these versions below. Conditional convergence in probability, together with boundedness, now yields

$$\mathbb{E}\big[|\,\mathrm{FDP} - D(z)| \mid Z = z\big] \longrightarrow 0 \quad \text{for almost every } z.$$

Dominated convergence and the tower property give

$$\mathbb{E}_{\Omega_N, Z, \varepsilon}\big[|\,\mathrm{FDP} - D(Z)|\big] \longrightarrow 0,$$

and a second application of the tower property yields (28). □

Lastly, we show that Σ can be made positive definite by an independent Gaussian perturbation. For a fixed realization $\omega = ((\theta_i, b_i))_{i=1}^N$, let $\xi \sim N(0, I_N)$ be independent and define

$$T_i^{(\eta)} = \theta_i + \sqrt{1-\eta^2}\left(b_i Z + \sqrt{1-b_i^2}\,\varepsilon_i\right) + \eta\xi_i, \quad 0 < \eta < 1. \tag{32}$$

Then

$$\Sigma_{\omega,\eta} = (1-\eta^2)\left[bb^\top + \mathrm{diag}(1-b_1^2, \dots, 1-b_N^2)\right] + \eta^2 I_N. \tag{33}$$

The following lemma shows that the perturbation makes the correlation matrix positive definite while preserving the FDR in the limit.

Lemma 2.9. *For every fixed ω,*

$$\mathrm{diag}(\Sigma_{\omega,\eta}) = \mathbf{1}_N, \quad \Sigma_{\omega,\eta} \succeq \eta^2 I_N, \tag{34}$$

and

$$\mathrm{FDR}_\omega^{(\eta)}(\mathrm{BH}_q) \longrightarrow \mathrm{FDR}_\omega(\mathrm{BH}_q) \quad (\eta \downarrow 0). \tag{35}$$

Proof. Equation (34) is immediate from (33). Under the coupling (32), $T^{(\eta)} \to T$ almost surely. Define $P_i^{(\eta)} = p(|T_i^{(\eta)}|)$. Then $P_i^{(\eta)} \to P_i = p(|T_i|)$ almost surely. Each P_i has a continuous marginal law, so

$$\mathbb{P}\left\{P_i \in \left\{\frac{qk}{N} : 1 \le k \le N\right\} \text{ for some } i\right\} = 0.$$

Off this event, the truth values of all finitely many comparisons $P_i^{(\eta)} \le qk/N$ agree with those of $P_i \le qk/N$ for all sufficiently small η. Hence the BH rejection count and rejected-null set are eventually constant. Bounded convergence proves (35). □

2.5 Construction of $\nu_{q,r}$

We construct a family of laws $\nu_{q,r}$ indexed by $r \downarrow 0$. For $a, y \ge 0$, put

$$L_a(y) = e^{-a^2/2}\cosh(ay).$$

For $y > 1$, let

$$a(y) = \arg\max_{\alpha \ge 0} L_\alpha(y), \quad y(\alpha) = a^{-1}(\alpha), \quad \alpha > 0.$$

Set

$$a_q = a(y_q). \tag{36}$$

Appendix A establishes the properties of these functions that we use below.

The quantities $b_r, \tau_r, \mu_r, \pi_{0,r}$, and c_r are defined below. The law $\nu_{q,r}$ has the three components displayed in Table 3.

Table 3: Two-sided population construction defining $\nu_{q,r}$.

Type	Mass	Θ	B
Nulls	$\pi_{0,r}$	0	r
Primary non-nulls	r	$c_r + y(b_r)$	1
Secondary non-nulls	$\tau_r \mu_r(\,\mathrm{d}a)$	$a/r + y(a)$	$1,\ a \in [b_r, a_q]$

Equivalently, for every bounded measurable $f : \mathbb{R} \times [-1,1] \to \mathbb{R}$,

$$\int f \, \mathrm{d}\nu_{q,r} = \pi_{0,r} f(0,r) + r f(c_r + y(b_r), 1) + \tau_r \int_{b_r}^{a_q} f\left(\frac{a}{r} + y(a), 1\right) \mu_r(\,\mathrm{d}a). \tag{37}$$

Definition of secondary non-nulls. For small r, set

$$b_r = r\, p^{-1}(r^2).$$

For the remaining definitions, take r small enough that $0 < b_r < a_q$. For $a \in [b_r, a_q]$, define

$$W_r(a) = \frac{[1 - qM\{y(a)\}]p(a/r)}{q}. \tag{38}$$

By Lemma A.1 and (289), the factor $1 - qM\{y(a)\}$ is nonnegative and decreasing on $[b_r, a_q]$; the factor $p(a/r)$ is decreasing in a. Hence W_r is continuous and decreasing, with $W_r(a_q) = 0$. Set

$$\tau_r := W_r(b_r) = \frac{[1 - qM\{y(b_r)\}]r^2}{q} = O(r^2). \tag{39}$$

We therefore call this the secondary non-null block; its total mass is $O(r^2) = o(r)$. Let μ_r be the probability measure on $[b_r, a_q]$ determined by

$$\mu_r([a, a_q]) = \frac{W_r(a)}{\tau_r}, \quad b_r \le a \le a_q. \tag{40}$$

In particular, $\mu_r([b_r, a_q]) = W_r(b_r)/\tau_r = 1$.

Definition of nulls. Put

$$\pi_{0,r} = 1 - r - \tau_r.$$

Since $\pi_{0,r} + r + \tau_r = 1$, the masses in (37) define a probability measure.

Definition of primary non-nulls. For all sufficiently small r, $\pi_{0,r} > 0$. Define c_r by

$$p(c_r) = \frac{q(r + \tau_r)}{1 - q\pi_{0,r}}. \tag{41}$$

The numerator is positive. Since $\tau_r = O(r^2)$ and $\pi_{0,r} = 1 - r - \tau_r$,

$$1 - q\pi_{0,r} = 1 - q + qr + q\tau_r \longrightarrow 1 - q > 0, \quad q(r + \tau_r) \longrightarrow 0.$$

Thus the right side of (41) is positive and smaller than one for small r, so c_r is well-defined.

2.6 Population FDR analysis under $\nu_{q,r}$

2.6.1 Preliminaries

Apply Subsection 2.3 with $\nu_q = \nu_{q,r}$. Denote the resulting quantities by

$$R_{r,z}, \quad R_{0,r,z}, \quad \bar{R}_{r,z}, \quad \bar{R}_{0,r,z}, \quad C_r(z).$$

For $c \ge u_q$,

$$\bar{R}_{r,z}(c) = \bar{R}_{0,r,z}(c) + \bar{R}_{1,r,z}(c) + \bar{R}_{2,r,z}(c), \tag{42}$$

where

$$
\begin{aligned}
\bar{R}_{0,r,z}(c) &= q\pi_{0,r}\frac{R_{0,r,z}\{p(c)\}}{p(c)}, \\
\bar{R}_{1,r,z}(c) &= \frac{qr}{p(c)}\mathbf{1}\{|c_r + y(b_r) + z| \geq c\}, \\
\bar{R}_{2,r,z}(c) &= \frac{q\tau_r}{p(c)}\int_{b_r}^{a_q}\mathbf{1}\left\{\left|\frac{a}{r} + y(a) + z\right| \geq c\right\}\mu_r(\,\mathrm{d}a).
\end{aligned}
\tag{43}
$$

For a measurable definition at every z, put

$$
C_r^\circ(z) = \inf\{c \in \mathbb{Q} : c \geq u_q,\ \bar{R}_{r,z}(c) > 1\}
$$

and define

$$
D_r(z) = \begin{cases} \min[1, \bar{R}_{0,r,z}\{C_r^\circ(z)\}], & C_r^\circ(z) < \infty, \\ 0, & C_r^\circ(z) = \infty. \end{cases}
$$

The joint Borel measurability of $(z,c) \mapsto \bar{R}_{r,z}(c)$ and $(z,c) \mapsto \bar{R}_{0,r,z}(c)$ makes both C_r° and D_r Borel measurable. At every regular crossing, the rational-cutoff argument in the proof of Theorem 2.8 gives $C_r^\circ(z) = C_r(z)$, and $\bar{R}_{0,r,z}\{C_r(z)\} \leq 1$. Hence

$$
D_r(z) = \bar{R}_{0,r,z}\{C_r(z)\}. \tag{44}
$$

Moreover, $0 \leq D_r(z) \leq 1$ for every z.

We next state two asymptotic lemmas that will be applied in the following analysis. The first gives the order of b_r and c_r as $r \downarrow 0$.

Lemma 2.10. *As $r \downarrow 0$,*

$$
b_r \downarrow 0, \quad \frac{b_r}{r} = p^{-1}(r^2) \to \infty, \quad p(b_r/r) = r^2, \quad y(b_r) \downarrow 1. \tag{45}
$$

Moreover,

$$
p(c_r) \sim \frac{qr}{1-q}, \quad c_r \to \infty, \quad rc_r \to 0, \quad \frac{b_r}{r} - c_r \to \infty. \tag{46}
$$

Proof. The identity $p(b_r/r) = r^2$ follows directly from $b_r = r\,p^{-1}(r^2)$. Applying (298) in Lemma A.2 gives

$$
\frac{b_r}{r} = p^{-1}(r^2) \sim \sqrt{4\log(1/r)} \to \infty, \quad b_r \sim 2r\sqrt{\log(1/r)} \to 0.
$$

The last limit in (45) follows from Lemma A.1: $a(y)$ is continuous and strictly increasing from $(1,\infty)$ onto $(0,\infty)$, with $\lim_{y\downarrow 1} a(y) = 0$. Since $y(\cdot)$ is its inverse and $b_r \downarrow 0$, we have $y(b_r) \downarrow 1$.

By (41) and (39),

$$
p(c_r) = \frac{q(r+\tau_r)}{1 - q\pi_{0,r}} \sim \frac{qr}{1-q}.
$$

For every fixed $\delta \in (0, q/(1-q))$, for all sufficiently small r,

$$
\left\{\frac{q}{1-q} - \delta\right\} r \leq p(c_r) \leq \left\{\frac{q}{1-q} + \delta\right\} r.
$$

Since p^{-1} is decreasing,

$$
p^{-1}\left(\left\{\frac{q}{1-q} + \delta\right\} r\right) \leq c_r \leq p^{-1}\left(\left\{\frac{q}{1-q} - \delta\right\} r\right).
$$

Equation (298) in Lemma A.2, applied to the two fixed positive constants, shows that both endpoints are asymptotic to $\sqrt{2\log(1/r)}$. Hence

$$c_r \sim \sqrt{2\log(1/r)}.$$

Hence $c_r \to \infty$ and $rc_r \to 0$. Combining this with

$$\frac{b_r}{r} = p^{-1}(r^2) \sim \sqrt{4\log(1/r)}$$

gives

$$\frac{b_r}{r} - c_r \sim \left(2-\sqrt{2}\right)\sqrt{\log(1/r)} \longrightarrow \infty.$$

□

The next lemma gives a uniform approximation to the normalized conditional FDP contribution from nulls.

Lemma 2.11. *For every $A > 0$ and $T \geq 0$, as $r \downarrow 0$,*

$$\sup_{|z|\leq T}\sup_{0\leq c\leq A/r}\left|\frac{R_{0,r,z}\{p(c)\}}{p(c)} - L_{rc}(|z|)\right| \longrightarrow 0. \tag{47}$$

Proof. For $0 < r < 1$, the null pair $(\Theta, B) = (0, r)$ gives

$$R_{0,r,z}\{p(c)\} = \mathbb{P}\{|rz + \sqrt{1-r^2}\,\varepsilon| \geq c\}.$$

Write

$$x_\pm = \frac{c \mp rz}{\sqrt{1-r^2}}.$$

Then

$$R_{0,r,z}\{p(c)\} = \overline{\Phi}(x_+) + \overline{\Phi}(x_-).$$

Fix $C > 0$. Uniformly for $|z| \leq T$ and $C \leq c \leq A/r$,

$$\begin{aligned} x_\pm - c &= c\left\{\frac{1}{\sqrt{1-r^2}} - 1\right\} \mp \frac{rz}{\sqrt{1-r^2}} \\ &= \frac{cr^2}{\sqrt{1-r^2}\{1+\sqrt{1-r^2}\}} \mp \frac{rz}{\sqrt{1-r^2}}. \end{aligned}$$

Therefore,

$$|x_\pm - c| \leq \frac{Ar}{\sqrt{1-r^2}\{1+\sqrt{1-r^2}\}} + \frac{Tr}{\sqrt{1-r^2}} = O_{A,T}(r).$$

Consequently, for all sufficiently small r, the shifts $x_\pm - c$ are uniformly bounded and $x_\pm \geq C/2$. The uniform expansion (297) in Lemma A.2 therefore gives remainders $\rho_{r,C,\pm}(c,z)$ such that

$$\begin{aligned} \frac{\overline{\Phi}(x_\pm)}{\overline{\Phi}(c)} &= \frac{c}{x_\pm}\exp\left\{-\frac{x_\pm^2 - c^2}{2}\right\}\{1 + \rho_{r,C,\pm}(c,z)\}, \\ -\frac{x_\pm^2 - c^2}{2} &= -\frac{(rc)^2}{2} \pm rcz + O_{A,T}(r^2). \end{aligned} \tag{48}$$

and

$$\lim_{C\to\infty} \limsup_{r\downarrow 0} \sup_{|z|\le T} \sup_{C\le c\le A/r} \max_{\pm} |\rho_{r,C,\pm}(c,z)| = 0. \tag{49}$$

For each fixed $C > 0$, $c/x_{\pm} \to 1$ uniformly for $|z| \le T$ and $C \le c \le A/r$ as $r \downarrow 0$. To identify the leading term, put $t = rc$, so that $0 \le t \le A$. Then

$$\frac{1}{2}\left[\exp\left\{-\frac{t^2}{2}+tz\right\}+\exp\left\{-\frac{t^2}{2}-tz\right\}\right] = e^{-t^2/2}\cosh(tz) = L_t(|z|).$$

Because t and z range over compact sets, these exponential factors are uniformly bounded. Thus, after summing the two tails in (48) and using $p(c) = 2\overline{\Phi}(c)$, the errors from $c/x_{\pm}-1$ and from the $O_{A,T}(r^2)$ term vanish uniformly for each fixed C. Consequently, there is a constant $K_{A,T} < \infty$, independent of C, such that, for

$$\eta(C) = \limsup_{r\downarrow 0} \sup_{|z|\le T} \sup_{C\le c\le A/r} \left|\frac{R_{0,r,z}\{p(c)\}}{p(c)} - L_{rc}(|z|)\right|,$$

we have

$$\eta(C) \le K_{A,T} \limsup_{r\downarrow 0} \sup_{|z|\le T} \sup_{C\le c\le A/r} \max_{\pm} |\rho_{r,C,\pm}(c,z)|.$$

The right side tends to zero as $C \to \infty$ by (49). Hence $\eta(C) \to 0$.

It remains to control $0 \le c \le C$. On this range, $p(c) \ge p(C) > 0$. Also, uniformly over $0 \le c \le C$ and $|z| \le T$,

$$R_{0,r,z}\{p(c)\} = \overline{\Phi}\left(\frac{c-rz}{\sqrt{1-r^2}}\right) + \overline{\Phi}\left(\frac{c+rz}{\sqrt{1-r^2}}\right) = p(c) + O_{C,T}(r),$$

because the two arguments differ from c by $O_{C,T}(r)$ and $\overline{\Phi}$ has bounded derivative. Hence

$$\sup_{|z|\le T} \sup_{0\le c\le C} \left|\frac{R_{0,r,z}\{p(c)\}}{p(c)} - 1\right| \longrightarrow 0.$$

Moreover, $L_{rc}(|z|) = e^{-(rc)^2/2}\cosh(rc|z|) \to 1$ uniformly for $|z| \le T$ and $0 \le c \le C$. Thus

$$\limsup_{r\downarrow 0} \sup_{|z|\le T} \sup_{0\le c\le A/r} \left|\frac{R_{0,r,z}\{p(c)\}}{p(c)} - L_{rc}(|z|)\right| \le \eta(C).$$

Letting $C \to \infty$ proves the claim. □

The following lemma provides a generic criterion for deriving the limit of $C_r(z)$.

Lemma 2.12. *Let K be compact. For $z \in K$, let $m_r(z) > u_q$ and $w_r(z) > 0$. Suppose that, for every fixed $H > 0$,*

$$\inf_{z\in K}\{m_r(z) - Hw_r(z)\} > u_q \tag{50}$$

for all sufficiently small r, and

$$\sup_{z\in K} \sup_{|h|\le H} \left|\bar{R}_{r,z}\{m_r(z)+w_r(z)h\} - d(z) - \{1-d(z)\}e^{\xi(z)h}\right| \longrightarrow 0, \tag{51}$$

where $d : K \to [0,1)$ and $\xi : K \to (0,\infty)$ are continuous. Suppose also that, for every $h_0 > 0$,

$$\limsup_{r\downarrow 0} \sup_{z\in K} \sup_{u_q\le c\le m_r(z)-w_r(z)h_0} \bar{R}_{r,z}(c) < 1. \tag{52}$$

Then

$$\sup_{z\in K} \left|\frac{C_r(z)-m_r(z)}{w_r(z)}\right| \longrightarrow 0. \tag{53}$$

Proof. Fix $h_0 > 0$. Compactness gives

$$\inf_{z\in K}\{1-d(z)\}\{e^{\xi(z)h_0}-1\} > 0.$$

Hence (51) gives $\bar{R}_{r,z}\{m_r(z)+w_r(z)h_0\} > 1$ uniformly on K, while (52) excludes a crossing at or before $m_r(z)-w_r(z)h_0$. Thus

$$m_r(z)-w_r(z)h_0 < C_r(z) \le m_r(z)+w_r(z)h_0$$

uniformly on K. Letting $h_0 \downarrow 0$ proves (53). □

The next lemma gives regularity and the limiting null contribution once the localized cutoff has been identified.

Lemma 2.13. *Under the assumptions and notation of Lemma 2.12, assume in addition that, for every fixed $H > 0$,*

$$\sup_{z\in K}\sup_{|h|\le H}\left|\bar{R}_{0,r,z}\{m_r(z)+w_r(z)h\}-d(z)\right| \longrightarrow 0. \tag{54}$$

If, for some $H_* > 0$ and all sufficiently small r, the map

$$h \longmapsto \bar{R}_{r,z}\{m_r(z)+w_r(z)h\}$$

is differentiable with positive derivative on $(-H_*, H_*)$, and the map $h \mapsto \bar{R}_{0,r,z}\{m_r(z)+w_r(z)h\}$ is continuous there for every $z \in K$, then $C_r(z)$ is regular and

$$\sup_{z\in K}|D_r(z)-d(z)| \longrightarrow 0. \tag{55}$$

Proof. For all sufficiently small r, (53) gives

$$m_r(z)-w_r(z)H_* < C_r(z) < m_r(z)+w_r(z)H_*.$$

With $c = m_r(z)+w_r(z)h$, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}h}\bar{R}_{r,z}\{m_r(z)+w_r(z)h\} = w_r(z)\bar{R}'_{r,z}(c).$$

Since $w_r(z) > 0$,

$$0 < \frac{\mathrm{d}}{\mathrm{d}h}\bar{R}_{r,z}\{m_r(z)+w_r(z)h\} = w_r(z)\bar{R}'_{r,z}(c) \quad\Longrightarrow\quad \bar{R}'_{r,z}(c) > 0.$$

Thus

$$\bar{R}'_{r,z}(c) > 0, \qquad m_r(z)-w_r(z)H_* < c < m_r(z)+w_r(z)H_*. \tag{56}$$

Equations (53) and (50), together with (56), verify the hypotheses of Proposition 2.5, which gives regularity. Finally, put

$$h_r(z) = \frac{C_r(z)-m_r(z)}{w_r(z)}.$$

Equation (53) gives $h_r(z) \to 0$ uniformly. By (44),

$$D_r(z) = \bar{R}_{0,r,z}\{C_r(z)\} = \bar{R}_{0,r,z}\{m_r(z)+w_r(z)h_r(z)\}.$$

Evaluating (54) at $h = h_r(z)$ therefore proves (55). □

The next lemma derives the derivatives of the null and secondary non-null contributions.

Lemma 2.14. *For $c > 0$, put*

$$x_\pm = \frac{c \mp rz}{\sqrt{1-r^2}}, \qquad Q_\pm(c,z) = \frac{\overline{\Phi}(x_\pm)}{\overline{\Phi}(c)}.$$

The null contribution $\bar{R}_{0,r,z}(c)$ satisfies

$$\bar{R}'_{0,r,z}(c) = \frac{q\pi_{0,r}}{2} \sum_{\sigma\in\{-,+\}} Q_\sigma(c,z) \left\{ \lambda(c) - \frac{\lambda(x_\sigma)}{\sqrt{1-r^2}} \right\}, \tag{57}$$

$$\frac{\mathrm{d}}{\mathrm{d}c} \log \bar{R}_{0,r,z}(c) = \lambda(c) - \frac{Q_+(c,z)\lambda(x_+) + Q_-(c,z)\lambda(x_-)}{\sqrt{1-r^2}\{Q_+(c,z)+Q_-(c,z)\}}. \tag{58}$$

Define

$$g_r(a) = a + ry(a). \tag{59}$$

For $0 < a < a_q$,

$$g_r'(a) = 1 + ry'(a) > 1,$$

so g_r is strictly increasing on $[b_r, a_q]$. Finally, fix $y > 1$ and $c > 0$, and suppose that

$$g_r(b_r) > r(y-c). \tag{60}$$

If

$$g_r(b_r) < r(c+y) < g_r(a_q),$$

let $\beta_{r,y}(c) \in (b_r, a_q)$ be the unique solution of

$$g_r\{\beta_{r,y}(c)\} = r(c+y). \tag{61}$$

Conditional on $Z = -y$, a secondary non-null indexed by $a \in [b_r, a_q]$ is then rejected at cutoff c if and only if

$$a \in [\beta_{r,y}(c), a_q]. \tag{62}$$

Whenever $1 - qM\{y(\beta_{r,y}(c))\} > 0$, the secondary non-null contribution $\bar{R}_{2,r,-y}(c)$ satisfies

$$\frac{\partial \beta_{r,y}(c)}{\partial c} = \frac{r}{1 + ry'\{\beta_{r,y}(c)\}}, \tag{63}$$

$$\frac{\mathrm{d}}{\mathrm{d}c} \log \bar{R}_{2,r,-y}(c) = -\frac{qM'\{y(\beta_{r,y}(c))\}y'\{\beta_{r,y}(c)\}}{1 - qM\{y(\beta_{r,y}(c))\}} \frac{r}{1+ry'\{\beta_{r,y}(c)\}} - \frac{\lambda\{\beta_{r,y}(c)/r\}}{1+ry'\{\beta_{r,y}(c)\}} + \lambda(c), \tag{64}$$

$$\bar{R}'_{2,r,-y}(c) = \bar{R}_{2,r,-y}(c) \frac{\mathrm{d}}{\mathrm{d}c} \log \bar{R}_{2,r,-y}(c). \tag{65}$$

If $r(c+y) \le g_r(b_r)$, the secondary non-null block is fully rejected. Under the strict inequality $r(c+y) < g_r(b_r)$,

$$\bar{R}_{2,r,-y}(c) = \frac{q\tau_r}{p(c)}, \qquad \bar{R}'_{2,r,-y}(c) = -\frac{q\tau_r p'(c)}{p(c)^2} = \lambda(c)\bar{R}_{2,r,-y}(c).$$

If $r(c+y) \geq g_r(a_q)$, the secondary non-null contribution vanishes: at equality the rejected set is $\{a_q\}$, which has μ_r-measure zero by (40) and $W_r(a_q) = 0$. Under the strict inequality $r(c+y) > g_r(a_q)$,

$$\bar{R}_{2,r,-y}(c) = \bar{R}'_{2,r,-y}(c) = 0.$$

As $r(c+y) \downarrow g_r(b_r)$, $\beta_{r,y}(c) \downarrow b_r$; as $r(c+y) \uparrow g_r(a_q)$, $\beta_{r,y}(c) \uparrow a_q$. Since W_r is continuous, $W_r(b_r) = \tau_r$, and $W_r(a_q) = 0$, the interior expression

$$\bar{R}_{2,r,-y}(c) = \frac{qW_r\{\beta_{r,y}(c)\}}{p(c)}$$

matches the endpoint values. Thus the contribution is continuous at both equalities. Its one-sided derivatives need not agree there, so a two-sided derivative is not asserted.

Proof. Since

$$\frac{\mathrm{d}x_\pm}{\mathrm{d}c} = \frac{1}{\sqrt{1-r^2}}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\log\overline{\Phi}(t) = -\lambda(t),$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}c}\log Q_\pm(c,z) = -\frac{\lambda(x_\pm)}{\sqrt{1-r^2}} + \lambda(c).$$

Multiplying by $Q_\pm(c,z)$ and summing gives

$$\begin{aligned}\bar{R}'_{0,r,z}(c) &= \frac{q\pi_{0,r}}{2}\{Q'_+(c,z) + Q'_-(c,z)\}\\ &= \frac{q\pi_{0,r}}{2}\sum_{\sigma\in\{-,+\}} Q_\sigma(c,z)\left\{\lambda(c) - \frac{\lambda(x_\sigma)}{\sqrt{1-r^2}}\right\},\end{aligned}$$

which is (57), and

$$\begin{aligned}\frac{\bar{R}'_{0,r,z}(c)}{\bar{R}_{0,r,z}(c)} &= \frac{Q'_+(c,z) + Q'_-(c,z)}{Q_+(c,z) + Q_-(c,z)}\\ &= \lambda(c) - \frac{Q_+(c,z)\lambda(x_+) + Q_-(c,z)\lambda(x_-)}{\sqrt{1-r^2}\{Q_+(c,z) + Q_-(c,z)\}},\end{aligned}$$

which is (58).

For a secondary non-null indexed by a, conditional on $Z = -y$,

$$T(a) = \frac{a}{r} + y(a) - y, \qquad r\{T(a) + y\} = g_r(a).$$

By (59) and the positivity of y' in Lemma A.1, $g'_r(a) = 1 + ry'(a) > 1$, proving the asserted strict monotonicity. Equation (60) gives

$$g_r(a) \geq g_r(b_r) > r(y-c) \implies T(a) > -c,$$

so rejection can occur only through $T(a) \geq c$. In the interior case, (61) and strict monotonicity give

$$T(a) \geq c \iff g_r(a) \geq r(c+y) \iff a \geq \beta_{r,y}(c).$$

This proves (62). Thus

$$\begin{aligned}\bar{R}_{2,r,-y}(c) &= \frac{q\tau_r}{p(c)}\mu_r([\beta_{r,y}(c), a_q]) \\ &= \frac{qW_r\{\beta_{r,y}(c)\}}{p(c)} \\ &= [1 - qM\{y(\beta_{r,y}(c))\}]\frac{p\{\beta_{r,y}(c)/r\}}{p(c)},\end{aligned}$$

by (40) and (38).

Differentiating the threshold equation yields

$$[1 + ry'\{\beta_{r,y}(c)\}]\frac{\partial\beta_{r,y}(c)}{\partial c} = r,$$

which proves (63). Since $p'(t)/p(t) = -\lambda(t)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}c}\log\bar{R}_{2,r,-y}(c) = &-\frac{qM'\{y(\beta_{r,y}(c))\}y'\{\beta_{r,y}(c)\}}{1 - qM\{y(\beta_{r,y}(c))\}}\frac{\partial\beta_{r,y}(c)}{\partial c} \\ &- \lambda\{\beta_{r,y}(c)/r\}\frac{1}{r}\frac{\partial\beta_{r,y}(c)}{\partial c} + \lambda(c).\end{aligned}$$

Substitution of (63) gives (64), and

$$\bar{R}'_{2,r,-y}(c) = \bar{R}_{2,r,-y}(c)\frac{\mathrm{d}}{\mathrm{d}c}\log\bar{R}_{2,r,-y}(c)$$

gives (65). The same equivalence $T(a) \geq c \iff g_r(a) \geq r(c+y)$ gives the endpoint cases:

$$\bar{R}_{2,r,-y}(c) = \begin{cases} q\tau_r/p(c), & r(c+y) \leq g_r(b_r), \\ 0, & r(c+y) \geq g_r(a_q), \end{cases} \qquad \bar{R}'_{2,r,-y}(c) = \begin{cases} \lambda(c)\bar{R}_{2,r,-y}(c), & r(c+y) < g_r(b_r), \\ 0, & r(c+y) > g_r(a_q). \end{cases}$$

Because W_r is continuous, (40) implies that μ_r has no atoms. Thus the displayed endpoint values agree with the limits from the interior, so the contribution is continuous at both thresholds. □

2.6.2 The regime $z > -1$

Lemma 2.15. *For every compact $K \subset (-1, \infty)$ and every $H > 0$, uniformly for $z \in K$ and $|h| \leq H$,*

$$\bar{R}_{r,z}\left(c_r + \frac{h}{c_r}\right) \longrightarrow q + (1-q)e^h. \tag{66}$$

Moreover, for every $h_0 > 0$,

$$\limsup_{r\downarrow 0}\sup_{z\in K}\sup_{u_q\leq c\leq c_r - h_0/c_r}\bar{R}_{r,z}(c) \leq q + (1-q)e^{-h_0} < 1. \tag{67}$$

Consequently,

$$C_r(z) = c_r + o(c_r^{-1}) \tag{68}$$

and

$$D_r(z) \longrightarrow q, \tag{69}$$

uniformly on K. Moreover, for sufficiently small r, $C_r(z)$ is regular for every $z \in K$.

Proof. **Proof of** (66)**.** Fix $H > 0$, take $z \in K$, and take $|h| \le H$. The primary non-null has $B = 1$ and $\Theta = c_r + y(b_r)$, so conditional on $Z = z$ its value is $c_r + y(b_r) + z$. Since $K \subset (-1, \infty)$ is compact,

$$\eta_K := \inf_{z \in K}(1 + z) > 0.$$

By Lemma 2.10, $c_r \to \infty$ and $y(b_r) \downarrow 1$. For the fixed H, this implies $H/c_r \to 0$. Combining these limits, for sufficiently small r,

$$|y(b_r) - 1| \le \frac{\eta_K}{4}, \quad \frac{H}{c_r} \le \frac{\eta_K}{4}. \tag{70}$$

For $z \in K$ and $|h| \le H$, the definition of η_K gives $z \ge -1 + \eta_K$, while (70) gives $y(b_r) \ge 1 - \eta_K/4$ and $-h/c_r \ge -H/c_r \ge -\eta_K/4$. Therefore

$$y(b_r) + z - \frac{h}{c_r} \ge \left(1 - \frac{\eta_K}{4}\right) + (-1 + \eta_K) - \frac{\eta_K}{4} = \frac{\eta_K}{2}, \quad z \in K, \ |h| \le H.$$

Equivalently, $y(b_r) + z \ge h/c_r + \eta_K/2$. Hence the conditional value of the primary non-null satisfies

$$\begin{aligned} |c_r + y(b_r) + z| &= c_r + y(b_r) + z \\ &= c_r + \frac{h}{c_r} + \left\{ y(b_r) + z - \frac{h}{c_r} \right\} \\ &\ge c_r + \frac{h}{c_r} + \frac{\eta_K}{2} > c_r + \frac{h}{c_r}, \end{aligned} \tag{71}$$

so the primary non-null is rejected at cutoff $c_r + h/c_r$, uniformly for $z \in K$ and $|h| \le H$.

Next consider a secondary non-null indexed by $a \in [b_r, a_q]$. Its conditional value at $Z = z$ is $a/r + y(a) + z$. Since $a \ge b_r$, we have $a/r \ge b_r/r$. Also, Lemma A.1 shows that $y(\cdot)$, the inverse of the strictly increasing map $a(\cdot)$, is increasing; hence $y(a) \ge y(b_r)$. Together with $z \ge \inf_{u \in K} u$ and $h \le H$, this gives

$$\frac{a}{r} + y(a) + z - \left(c_r + \frac{h}{c_r}\right) \ge \left(\frac{b_r}{r} - c_r\right) + y(b_r) + \inf_{u \in K} u - \frac{H}{c_r}. \tag{72}$$

Here $\frac{b_r}{r} - c_r \to \infty$ by (46), while

$$y(b_r) + \inf_{u \in K} u - \frac{H}{c_r} \longrightarrow 1 + \inf_{u \in K} u \tag{73}$$

because $y(b_r) \downarrow 1$ and $H/c_r \to 0$. Equations (46) and (73) show that the right side of (72) tends to $+\infty$. Hence it is positive for sufficiently small r, uniformly in $z \in K$, $|h| \le H$, and $a \in [b_r, a_q]$. Thus every secondary non-null is rejected at cutoff $c_r + h/c_r$.

Lastly, we analyze the nulls. Since $|h| \le H$ and $rc_r \to 0$ by (46),

$$r\left(c_r + \frac{h}{c_r}\right) = rc_r + \frac{rh}{c_r} \longrightarrow 0. \tag{74}$$

Let $T_K = \sup_{z \in K} |z| < \infty$. Since $c_r \to \infty$ by (46) in Lemma 2.10, while (74) gives a vanishing rescaled cutoff, we have

$$u_q \le c_r + \frac{h}{c_r} \le \frac{1}{r}$$

for all sufficiently small r, uniformly for $z \in K$ and $|h| \le H$. Applying (47) in Lemma 2.11 with $A = 1$ and $T = T_K$ gives

$$\frac{R_{0,r,z}\{p(c_r + h/c_r)\}}{p(c_r + h/c_r)} = L_{r(c_r + h/c_r)}(|z|) + o(1).$$

Moreover,

$$L_{r(c_r+h/c_r)}(|z|) = \exp\left\{-\frac{r^2(c_r+h/c_r)^2}{2}\right\}\cosh(r(c_r+h/c_r)|z|) = 1+o(1),$$

uniformly for $z \in K$ and $|h| \le H$. Since $\pi_{0,r} = 1 - r - \tau_r \to 1$, it follows that

$$\bar{R}_{0,r,z}(c_r + h/c_r) = q + o(1).$$

By (41),

$$\frac{q(r+\tau_r)}{p(c_r+h/c_r)} = (1-q\pi_{0,r})\frac{p(c_r)}{p(c_r+h/c_r)} \longrightarrow (1-q)e^h$$

by (302) in Lemma A.2. This proves (66).

Proof of (67)**.** Fix $h_0 > 0$. Equation (71), evaluated at $h = h_0$, gives

$$|c_r + y(b_r) + z| > c_r + \frac{h_0}{c_r},$$

so the primary non-null is rejected at cutoff $c_r + h_0/c_r$. Equations (72), (46), and (73) give

$$\frac{a}{r} + y(a) + z > c_r + \frac{h_0}{c_r}, \quad a \in [b_r, a_q],$$

for sufficiently small r, uniformly in $z \in K$. Thus every secondary non-null is also rejected at cutoff $c_r + h_0/c_r$. Since rejection at cutoff c means $|T_i| \ge c$, all non-nulls are rejected for every $u_q \le c \le c_r + h_0/c_r$. Therefore, on this interval,

$$\bar{R}_{r,z}(c) = \bar{R}_{0,r,z}(c) + \frac{q(r+\tau_r)}{p(c)}.$$

For $u_q \le c \le c_r - h_0/c_r$, Lemma 2.11 and $rc_r \to 0$ from (46) give

$$\sup_{z\in K}\ \sup_{u_q \le c \le c_r - h_0/c_r} \bar{R}_{0,r,z}(c) \le q + o(1).$$

Also $p(c) \ge p(c_r - h_0/c_r)$, so (41) and (302) in Lemma A.2 give

$$\sup_{u_q \le c \le c_r - h_0/c_r} \frac{q(r+\tau_r)}{p(c)} \le \frac{q(r+\tau_r)}{p(c_r - h_0/c_r)} = (1-q\pi_{0,r})\frac{p(c_r)}{p(c_r-h_0/c_r)} \longrightarrow (1-q)e^{-h_0}.$$

Taking the supremum over $z \in K$ and then the limsup in r proves (67).

Proof of (68)**.** Apply Lemma 2.12 with

$$m_r(z) = c_r, \qquad w_r(z) = c_r^{-1}, \qquad d(z) = q, \qquad \xi(z) = 1.$$

The set K is compact by hypothesis. Since $c_r \to \infty$, for all sufficiently small r, $m_r(z) = c_r > u_q$ and $w_r(z) = c_r^{-1} > 0$. Moreover, $d \equiv q$ and $\xi \equiv 1$ are continuous on K, with $d(z) \in [0,1)$ and $\xi(z) > 0$. For every fixed $H > 0$,

$$\inf_{z\in K}\{m_r(z) - Hw_r(z)\} = c_r - \frac{H}{c_r} > u_q$$

for all sufficiently small r, which verifies (50). Under these choices, (51) becomes

$$\sup_{z\in K}\sup_{|h|\le H}\left|\bar{R}_{r,z}\left(c_r+\frac{h}{c_r}\right)-q-(1-q)e^h\right|\longrightarrow 0,$$

which is exactly (66). Its left-of-crossing assumption (52) becomes, for every $h_0>0$,

$$\limsup_{r\downarrow 0}\sup_{z\in K}\sup_{u_q\le c\le c_r-h_0/c_r}\bar{R}_{r,z}(c)<1.$$

This follows from (67), because $q+(1-q)e^{-h_0}<1$. Hence all assumptions of Lemma 2.12 hold, and (53) gives

$$\sup_{z\in K}\left|\frac{C_r(z)-c_r}{c_r^{-1}}\right|=\sup_{z\in K}c_r|C_r(z)-c_r|\longrightarrow 0,$$

which is (68) uniformly on K.

Proof of (69)**.** Fix $H>0$. On $|c-c_r|\le H/c_r$, all non-nulls are rejected, uniformly over $z\in K$, and hence

$$\bar{R}_{r,z}(c)=\bar{R}_{0,r,z}(c)+\frac{q(r+\tau_r)}{p(c)}. \tag{75}$$

In the notation of Lemma 2.14, uniformly for $|c-c_r|\le H/c_r$ and $z\in K$, (46) gives

$$c\longrightarrow\infty,\qquad rc\longrightarrow 0.$$

For all sufficiently small r, this range lies in $[C,1/r]$, with $C\to\infty$. Thus (48)–(49) in Lemma 2.11 give, for either sign,

$$\begin{aligned}Q_\pm(c,z)&=\frac{c}{x_\pm}\exp\left\{-\frac{(rc)^2}{2}\pm rcz+O(r^2)\right\}\{1+o(1)\}\\&=1+o(1),\end{aligned}$$

because $c/x_\pm\to 1$, $rc\to 0$, and z ranges over the compact set K. Also $x_\pm/c_r\to 1$, $c/c_r\to 1$, and $\sqrt{1-r^2}\to 1$. Thus (293) in Lemma A.2, applied directly to (57) in Lemma 2.14, gives

$$\sup_{z\in K}\sup_{|c-c_r|\le H/c_r}|\bar{R}'_{0,r,z}(c)|=o(c_r).$$

For $|c-c_r|\le H/c_r$, write $c=c_r+h/c_r$, where $|h|\le H$. By (41),

$$\frac{q(r+\tau_r)}{p(c)}=(1-q\pi_{0,r})\frac{p(c_r)}{p(c)},$$

and (302) in Lemma A.2 gives

$$\frac{p(c_r)}{p(c)}=\frac{p(c_r)}{p(c_r+h/c_r)}\longrightarrow e^h,$$

uniformly for $|h|\le H$. Since $e^h\ge e^{-H}$ on this range and $1-q\pi_{0,r}\to 1-q>0$, for all sufficiently small r,

$$\inf_{|c-c_r|\le H/c_r}\frac{q(r+\tau_r)}{p(c)}\ge\frac{1-q}{2}\cdot\frac{e^{-H}}{2}.$$

Thus we may take $m_H = (1-q)e^{-H}/4 > 0$. Since all non-nulls are fully rejected for $|c - c_r| \le H/c_r$, their combined contribution is $q(r+\tau_r)/p(c)$, whose derivative is $\lambda(c)q(r+\tau_r)/p(c)$. Together with $\lambda(c) \sim c_r$ uniformly for $|c - c_r| \le H/c_r$, this gives

$$\inf_{|c-c_r|\le H/c_r} \frac{\mathrm{d}}{\mathrm{d}c}\frac{q(r+\tau_r)}{p(c)} \ge \frac{m_H}{2}c_r$$

for all sufficiently small r. By contrast, $\bar{R}'_{0,r,z}(c) = o(c_r)$ uniformly over $z \in K$ and $|c - c_r| \le H/c_r$. Differentiating (75) therefore shows that

$$\inf_{z\in K} \inf_{|c-c_r|\le H/c_r} \bar{R}'_{r,z}(c) > 0 \tag{76}$$

for all sufficiently small r. By (68), uniformly for $z \in K$,

$$c_r - \frac{H}{c_r} < C_r(z) < c_r + \frac{H}{c_r}$$

for all sufficiently small r; moreover, $c_r - H/c_r > u_q$. Equation (76) and the smoothness of the null contribution on this interval therefore verify the hypotheses of Proposition 2.5, so $C_r(z)$ is regular.

Equation (44) gives

$$D_r(z) = \bar{R}_{0,r,z}\{C_r(z)\}.$$

Put $h_r(z) = c_r\{C_r(z) - c_r\}$. Equation (68) gives $h_r(z) \to 0$ uniformly on K. The null limit established in the proof of (66), evaluated at $h = h_r(z)$, now gives

$$\sup_{z\in K} |D_r(z) - q| = \sup_{z\in K} \left|\bar{R}_{0,r,z}\{C_r(z)\} - q\right| \longrightarrow 0,$$

which is (69). $\square$

2.6.3 The regime $z \in (-y_q, -1)$

Let $y = -z$. For $1 < y < y_q$, put

$$\kappa(y) = a(y)y'\{a(y)\} > 0. \tag{77}$$

The positivity follows from Lemma A.1, since $a(y) > 0$ and $y'\{a(y)\} > 0$ for $1 < y < y_q$. For the remainder of the population analysis, define

$$s_r(y) = c_r + y(b_r) - y. \tag{78}$$

The next lemma separates the primary non-null value from the cutoffs on the r^{-1} scale.

Lemma 2.16. *For every compact $I \subset (1,\infty)$ and every $H > 0$, if $a_I = \inf_{y\in I} a(y) > 0$, then*

$$\begin{aligned} \sup_{y\in I} r|s_r(y)| &\le rc_r + r\sup_{y\in I}|y(b_r) - y| \longrightarrow 0, \\ \inf_{\substack{y\in I\\|h|\le H}} r\left\{\frac{a(y)}{r} + h\right\} &= a_I - rH \longrightarrow a_I > 0. \end{aligned} \tag{79}$$

Consequently, the primary non-null is not rejected at cutoff $a(y)/r + h$ for all sufficiently small r, uniformly for $y \in I$ and $|h| \le H$.

Proof. Conditional on $Z = -y$, the primary non-null value is $s_r(y)$ by (78). The two limits in (79) follow from (45) and (46) in Lemma 2.10. Their limiting bounds are separated by $a_I > 0$, which proves the rejection claim. □

Lemma 2.17. *For every compact $I \subset (1, \infty)$,*

$$\sup_{y \in I} \sup_{u_q \le s \le s_r(y)} \bar{R}_{r,-y}(s) \le q + o(1) < 1. \tag{80}$$

Proof. Put $y_I = \inf I > 1$ and $\delta_I^{\text{shift}} = (y_I - 1)/2 > 0$. Since $y(b_r) \downarrow 1$,

$$\delta_I^{\text{shift}} \le y - y(b_r) \le \sup_{u \in I} u - 1 =: \Delta_I < \infty \tag{81}$$

for all sufficiently small r, uniformly for $y \in I$. Hence $s_r(y) \le c_r - \delta_I^{\text{shift}}$. From (42) and the bound by one on each non-null rejection probability,

$$\bar{R}_{r,-y}(s) \le \bar{R}_{0,r,-y}(s) + \frac{q(r + \tau_r)}{p(s)}, \qquad u_q \le s \le s_r(y).$$

Uniformly on this range, $0 \le rs \le rc_r \to 0$. Therefore (47) in Lemma 2.11 gives

$$\bar{R}_{0,r,-y}(s) = q + o(1).$$

Since p is decreasing, (41) yields

$$\begin{aligned} \frac{q(r + \tau_r)}{p(s)} &\le \frac{q(r + \tau_r)}{p\{s_r(y)\}} \\ &= (1 - q\pi_{0,r}) \frac{p(c_r)}{p\{c_r + y(b_r) - y\}}. \end{aligned}$$

Here $s_r(y) \to \infty$ uniformly and the shift $y - y(b_r)$ lies in $[\delta_I^{\text{shift}}, \Delta_I]$. Thus (303) in Lemma A.2 gives

$$\sup_{y \in I} \frac{p(c_r)}{p\{c_r + y(b_r) - y\}} \longrightarrow 0.$$

Combining the bounds proves (80). □

We first prove a lemma that is useful for the key result in this regime.

Lemma 2.18. *For every compact $I \subset (1, y_q)$ and every $H > 0$, there is an interval U_I whose closure is contained in $(0, a_q)$ and contains $\{a(y) : y \in I\}$. For all sufficiently small r, with g_r as in (59), the equation*

$$g_r\{\tilde{\alpha}_r(y, h)\} = g_r\{a(y)\} + rh \tag{82}$$

has a unique solution $\tilde{\alpha}_r(y, h) \in U_I$, uniformly for $y \in I$ and $|h| \le H$, and

$$\tilde{\alpha}_r(y, h) = a(y) + O(r). \tag{83}$$

Conditional on $Z = -y$, a secondary non-null with $a = \tilde{a}$ is rejected at cutoff $a(y)/r + h$ if and only if

$$\tilde{a} \in [\tilde{\alpha}_r(y, h), a_q].$$

Finally,

$$\log \frac{p\{\tilde{\alpha}_r(y, h)/r\}}{p(a(y)/r + h)} = \kappa(y)h + o(1), \tag{84}$$

uniformly for $y \in I$ and $|h| \le H$.

Proof. Fix a compact $I \subset (1, y_q)$ and $H > 0$ as in the statement. Let

$$A_I = \{a(y) : y \in I\}.$$

Because $I \subset (1, y_q)$ is compact, A_I is a compact subset of $(0, a_q)$. Choose an interval U_I whose closure is contained in $(0, a_q)$ and such that $A_I \subset U_I$. By (45) in Lemma 2.10, $b_r \to 0$, so $U_I \subset (b_r, a_q)$ for all sufficiently small r. Lemma 2.14 then shows that g_r is strictly increasing on U_I.

Put $d_I = \mathrm{dist}(A_I, U_I^c) > 0$. If $rH < d_I/2$, then $a(y) \pm rH \in U_I$ for all $y \in I$. Since $g_r' \geq 1$ on U_I, the intermediate value theorem gives a unique solution to (82) in U_I, and

$$|\tilde{\alpha}_r(y, h) - a(y)| \leq rH.$$

Hence, (83) is proved.

Put $c = a(y)/r + h$ and $\beta = \tilde{\alpha}_r(y, h)$. If $a_I = \inf_{u \in I} a(u) > 0$, then

$$r(y - c) = ry - a(y) - rh \leq r \sup_{u \in I} u - a_I + rH < 0 < g_r(b_r)$$

for all sufficiently small r, uniformly for $y \in I$ and $|h| \leq H$. Thus (60) holds uniformly. Moreover, (82) and $y\{a(y)\} = y$ give

$$g_r(\beta) = g_r\{a(y)\} + rh = r(c + y).$$

This is (61), so (62) shows that a secondary non-null is rejected exactly when its index lies in $[\tilde{\alpha}_r(y, h), a_q]$.

Write $\beta = \tilde{\alpha}_r(y, h)$. Differentiating (82) with respect to h gives

$$\frac{\partial \beta}{\partial h} = \frac{r}{1 + ry'(\beta)}.$$

Since $p'(x)/p(x) = -\lambda(x)$,

$$\frac{\partial}{\partial h} \log \frac{p(\beta/r)}{p\{a(y)/r + h\}} = \lambda\Big(\frac{a(y)}{r} + h\Big) - \frac{\lambda(\beta/r)}{1 + ry'(\beta)}. \tag{85}$$

Moreover, (82) and $y\{a(y)\} = y$ imply

$$\frac{\beta}{r} = \frac{a(y)}{r} + h + y - y(\beta). \tag{86}$$

The function $a(y)$ is bounded away from zero on I, and (83) gives $\beta = a(y) + O(r)$ uniformly. Hence both arguments of λ in (85) are of order r^{-1}. Using $\lambda(x) = x + O(x^{-1})$ from (293) in Lemma A.2 and then (86), we obtain

$$\begin{aligned}
\lambda\Big(\frac{a(y)}{r} + h\Big) - \frac{\lambda(\beta/r)}{1 + ry'(\beta)} &= \frac{a(y)}{r} + h - \frac{\beta/r}{1 + ry'(\beta)} + O(r) \\
&= \frac{\{a(y) + rh\}y'(\beta) + y(\beta) - y}{1 + ry'(\beta)} + O(r) \\
&\longrightarrow a(y)y'\{a(y)\} = \kappa(y)
\end{aligned}$$

uniformly for $y \in I$ and $|h| \leq H$. At $h = 0$, uniqueness in (82) gives $\beta = a(y)$, so the logarithm in (85) is zero. Integrating from 0 to h proves (84). $\square$

Lemma 2.19. *For every compact $I \subset (1, y_q)$ and every $H > 0$, uniformly for $y \in I$ and $|h| \leq H$,*

$$\bar{R}_{r,-y}\left(\frac{a(y)}{r} + h\right) \longrightarrow qM(y) + \{1 - qM(y)\}e^{\kappa(y)h}. \tag{87}$$

Moreover, for every $h_0 > 0$, there is $\gamma > 0$ such that, for all sufficiently small r,

$$\sup_{y \in I} \sup_{u_q \leq c \leq a(y)/r - h_0} \bar{R}_{r,-y}(c) \leq 1 - \gamma. \tag{88}$$

Consequently, uniformly for $y \in I$,

$$C_r(-y) - \frac{a(y)}{r} \longrightarrow 0 \tag{89}$$

and

$$D_r(-y) \longrightarrow qM(y). \tag{90}$$

Moreover, for sufficiently small r, $C_r(-y)$ is regular for every $y \in I$.

Proof. **Proof of (87).** Fix $H > 0$, take $y \in I$, and take $|h| \leq H$. By Lemma 2.18, a secondary non-null with $a = \tilde{a}$ is rejected at cutoff $a(y)/r + h$ if and only if

$$\tilde{a} \in [\tilde{\alpha}_r(y, h), a_q].$$

By (40), their mass under the secondary component is

$$\tau_r \mu_r([\tilde{\alpha}_r(y, h), a_q]) = W_r\{\tilde{\alpha}_r(y, h)\}.$$

The definition (38) therefore gives

$$\frac{qW_r\{\tilde{\alpha}_r(y, h)\}}{p(a(y)/r + h)} = [1 - qM\{y(\tilde{\alpha}_r(y, h))\}] \frac{p\{\tilde{\alpha}_r(y, h)/r\}}{p(a(y)/r + h)}.$$

By (83), $y(\tilde{\alpha}_r(y, h)) \to y$ uniformly for $y \in I$ and $|h| \leq H$. Equation (84) gives

$$\frac{p\{\tilde{\alpha}_r(y, h)/r\}}{p(a(y)/r + h)} \longrightarrow e^{\kappa(y)h},$$

uniformly for $y \in I$ and $|h| \leq H$. Hence

$$\frac{qW_r\{\tilde{\alpha}_r(y, h)\}}{p(a(y)/r + h)} = [1 - qM\{y(\tilde{\alpha}_r(y, h))\}] \frac{p\{\tilde{\alpha}_r(y, h)/r\}}{p(a(y)/r + h)} \longrightarrow \{1 - qM(y)\}e^{\kappa(y)h} \tag{91}$$

Lemma 2.16, through (79), shows that the primary non-null is not rejected at cutoff $a(y)/r + h$, uniformly for $y \in I$ and $|h| \leq H$.

Since $a(y)$ is bounded away from zero on I and $a(y) \leq a_q$, for all sufficiently small r,

$$u_q \leq \frac{a(y)}{r} + h \leq \frac{a_q + 1}{r}$$

uniformly for $y \in I$ and $|h| \leq H$. Thus (47) in Lemma 2.11, applied with $A = a_q + 1$ and $T = \sup_{u \in I} u$, gives

$$\bar{R}_{0,r,-y}(a(y)/r + h) \longrightarrow qL_{a(y)}(y) = qM(y). \tag{92}$$

Combining (91) and (92) proves (87).

Proof of (88)**.** Lemma 2.17, through (80), excludes a crossing on $u_q \le s \le s_r(y)$, uniformly for $y \in I$. Fix $h_0 > 0$. The complementary range $s_r(y) < s \le a(y)/r - h_0$ is nonempty uniformly on I, because

$$\inf_{y\in I}\left\{\frac{a(y)}{r} - h_0 - s_r(y)\right\} \ge \frac{\inf_{y\in I} a(y)}{r} - c_r - O_I(1) \longrightarrow \infty,$$

where $rc_r \to 0$ by (46) in Lemma 2.10. It remains to prove that there is $\gamma > 0$ such that, for all sufficiently small r,

$$\sup_{y\in I}\ \sup_{s_r(y)<s\le a(y)/r-h_0} \bar{R}_{r,-y}(s) \le 1-\gamma \tag{93}$$

For $s > s_r(y)$, the identity $T_i = s_r(y) > 0$ gives $|T_i| = T_i < s$. For sufficiently small r, every secondary non-null value is positive uniformly over $y \in I$ and $\tilde{a} \in [b_r, a_q]$. For every $y > 1$ and $s > s_r(y)$, define

$$\mathcal{A}_{r,y,s} := \{\tilde{a} \in [b_r, a_q] : g_r(\tilde{a}) \ge r(s+y)\}, \qquad \beta_{r,y,s} := \inf(\mathcal{A}_{r,y,s} \cup \{a_q\}). \tag{94}$$

On the range $s_r(y) < s \le a(y)/r - h_0$, the set is nonempty for sufficiently small r, since $\tilde{a} = a_q$ satisfies the inequality uniformly over $y \in I$. By (59) and Lemma 2.14, g_r is strictly increasing. Thus a secondary non-null with $a = \tilde{a}$ is rejected if and only if $\tilde{a} \in [\beta_{r,y,s}, a_q]$. Therefore, by (42),

$$\bar{R}_{r,-y}(s) = \bar{R}_{0,r,-y}(s) + \frac{q\tau_r}{p(s)}\mu_r([\beta_{r,y,s}, a_q]). \tag{95}$$

Suppose (93) fails for this h_0. Then there are

$$r_n \downarrow 0, \quad y_n \in I, \quad s_{r_n}(y_n) < s_n \le a(y_n)/r_n - h_0, \quad \liminf_{n\to\infty} \bar{R}_{r_n,-y_n}(s_n) \ge 1. \tag{96}$$

After taking a subsequence,

$$y_n \to y \in I, \quad a(y_n) \to a(y), \quad x_n = r_n s_n \to x \in [0, a(y)]. \tag{97}$$

Define

$$h_n := s_n - \frac{a(y_n)}{r_n}.$$

Then $h_n \le -h_0$ by (96). If $h_n = O(1)$, then after passing to a further subsequence $h_n \to h \le -h_0$. Equation (87) gives

$$\bar{R}_{r_n,-y_n}(s_n) \longrightarrow qM(y) + \{1-qM(y)\}e^{\kappa(y)h} \le qM(y) + \{1-qM(y)\}e^{-\kappa(y)h_0} < 1,$$

contradicting $\liminf_{n\to\infty} \bar{R}_{r_n,-y_n}(s_n) \ge 1$. Hence, after passing to a further subsequence if necessary, it remains to consider

$$h_n \to -\infty. \tag{98}$$

We next identify the rejected secondary non-nulls at cutoff s_n. A secondary non-null with index $\tilde{a} \in [b_{r_n}, a_q]$ has $(\Theta, B) = (\tilde{a}/r_n + y(\tilde{a}), 1)$. Conditional on $Z = -y_n$, its value is

$$T_{\tilde{a},n} = \frac{\tilde{a}}{r_n} + y(\tilde{a}) - y_n.$$

For all large n, this value is positive uniformly in $\tilde{a} \in [b_{r_n}, a_q]$, because $b_{r_n}/r_n \to \infty$ and $y(\tilde{a}) - y_n$ is bounded below. Hence $|T_{\tilde{a},n}| = T_{\tilde{a},n}$, and the rejection rule $|T_{\tilde{a},n}| \ge s_n$ is equivalent to

$$\frac{\tilde{a}}{r_n} + y(\tilde{a}) - y_n \ge s_n.$$

The rejected set is nonempty because the endpoint $\tilde{a} = a_q$ is rejected. Indeed, compactness of $I \subset (1, y_q)$ and continuity of $a(\cdot)$ give

$$\Delta_I := a_q - \sup_{u \in I} a(u) > 0.$$

Since $s_n \le a(y_n)/r_n - h_0$ by (96),

$$\frac{a_q}{r_n} + y_q - y_n - s_n \ge \frac{a_q - a(y_n)}{r_n} + y_q - y_n + h_0 > 0$$

for large n, because $a_q - a(y_n) \ge \Delta_I$ and $y_n \le \sup I < y_q$. Define

$$G_n(\tilde{a}) := g_{r_n}(\tilde{a}) - r_n(s_n + y_n), \quad \tilde{a} \in [b_{r_n}, a_q],$$

where g_r is defined in (59). Multiplying the rejection inequality by r_n, a secondary non-null with index $\tilde{a}$ is rejected if and only if

$$G_n(\tilde{a}) \ge 0.$$

By (59) and Lemma 2.14, G_n is continuous and strictly increasing on $[b_{r_n}, a_q]$ for all large n. Therefore a secondary non-null with $a = \tilde{a}$ is rejected if and only if $\tilde{a} \in [\beta_n, a_q]$, where

$$\beta_n := \inf \{\tilde{a} \in [b_{r_n}, a_q] : G_n(\tilde{a}) \ge 0\}. \tag{99}$$

Put $d_n := y_n - y(\beta_n)$. If $\beta_n > b_{r_n}$, then $G_n(\beta_n) = 0$. Thus, in this case,

$$\frac{\beta_n}{r_n} - s_n = d_n. \tag{100}$$

If $\beta_n = b_{r_n}$, then $G_n(\beta_n) \ge 0$, so (100) is replaced by

$$\frac{\beta_n}{r_n} - s_n \ge d_n. \tag{101}$$

Moreover, $s_n \to \infty$, because $s_n > s_{r_n}(y_n) = c_{r_n} + y(b_{r_n}) - y_n$, $c_{r_n} \to \infty$, $y(b_{r_n}) \to 1$, and I is compact. Also $d_n > 0$ eventually. If $\beta_n = b_{r_n}$, this follows from the definition of d_n and (81) in Lemma 2.17, which give

$$d_n = y_n - y(b_{r_n}) \ge \delta_I^{\text{shift}} > 0.$$

If $\beta_n > b_{r_n}$, then

$$g_{r_n}(\beta_n) = r_n(s_n + y_n) = g_{r_n}\{a(y_n)\} + r_n h_n \le g_{r_n}\{a(y_n)\} - r_n h_0.$$

The strict monotonicity of g_{r_n} therefore gives $\beta_n < a(y_n)$, and hence $d_n = y_n - y(\beta_n) > 0$. The null contribution term in (95) already has a limit along this subsequence:

$$q\pi_{0,r_n} \frac{R_{0,r_n,-y_n}\{p(s_n)\}}{p(s_n)} \longrightarrow qL_x(y) \le qM(y) < 1. \tag{102}$$

Indeed, (47) in Lemma 2.11 applies because $r_n s_n \to x$, $y_n \to y$, and $\pi_{0,r_n} \to 1$. The inequality follows from $M(y) = \sup_{\alpha \ge 0} L_\alpha(y)$, the strict monotonicity of M on $(1, \infty)$, $y \in I \subset (1, y_q)$, and $qM(y_q) = 1$. Equations (95) and (102), together with the threshold characterization in (99), imply

$$\bar{R}_{r_n,-y_n}(s_n) = qL_x(y) + o(1) + \frac{q\tau_{r_n}}{p(s_n)} \mu_{r_n}([\beta_n, a_q]). \tag{103}$$

Moreover, (40), (38), (289) in Lemma A.1, and the bound from (100)–(101) give

$$\frac{q\tau_{r_n}}{p(s_n)}\mu_{r_n}([\beta_n,a_q]) = \frac{qW_{r_n}(\beta_n)}{p(s_n)} = [1-qM\{y(\beta_n)\}]\frac{p(\beta_n/r_n)}{p(s_n)} \le \frac{p(s_n+d_n)}{p(s_n)}. \tag{104}$$

It remains only to prove that

$$s_n d_n \longrightarrow \infty. \tag{105}$$

Equations (104), (105), and (304) in Lemma A.2 imply

$$\frac{q\tau_{r_n}}{p(s_n)}\mu_{r_n}([\beta_n,a_q]) = o(1).$$

Substituting this into (103) gives

$$\bar{R}_{r_n,-y_n}(s_n) \longrightarrow qL_x(y) < 1,$$

which yields a contradiction to (96).

To prove (105), we now consider three cases under (96), (97), and (98).

Case (a): $0 < x < a(y)$. Suppose $x > 0$. Since $b_{r_n} \to 0$ by (45) in Lemma 2.10 and $r_n s_n \to x$, the endpoint case $\beta_n = b_{r_n}$ cannot occur for large n: multiplying (101) by r_n would give

$$b_{r_n} - r_n s_n \ge r_n\{y_n - y(b_{r_n})\},$$

whose left side tends to $-x < 0$, while the right side is nonnegative for large n. Hence (100) holds eventually. Multiplying it by r_n gives

$$\beta_n - r_n s_n = r_n d_n.$$

The factor d_n is bounded. Indeed, I is compact, so $|y_n| \le \sup_{u\in I}|u|$. Also $\beta_n \in [b_{r_n}, a_q]$, and $y(\cdot)$ is increasing by Lemma A.1; hence

$$y(b_{r_n}) \le y(\beta_n) \le y(a_q) = y_q. \tag{106}$$

Since $y(b_{r_n}) \downarrow 1$ by (45) in Lemma 2.10, the sequence $y(\beta_n)$ is bounded. Thus $r_n d_n \to 0$, so $\beta_n - r_n s_n \to 0$. Since $r_n s_n \to x$, we have $\beta_n \to x$. Since $x < a(y)$ and $y(\cdot)$ is strictly increasing by Lemma A.1, $y(x) < y$. Together, $y_n \to y$ from (97) and $\beta_n \to x$ imply, by continuity of $y(\cdot)$, that $d_n \to d := y - y(x) > 0$. Also, (100) gives $\beta_n/r_n = s_n + d_n$. Therefore, for all large n,

$$\frac{\beta_n}{r_n} = s_n + d_n, \quad 0 < \frac{d}{2} \le d_n \le 2d, \quad s_n \to \infty.$$

As a result, (105) is proved.

Case (b): $x = 0$. Then $r_n s_n \to 0$. Since

$$s_n > s_{r_n}(y_n) = c_{r_n} + y(b_{r_n}) - y_n,$$

(46) gives $c_{r_n} \to \infty$, while (45) gives $y(b_{r_n}) \downarrow 1$ and compactness of I keeps y_n bounded. Hence $s_n \to \infty$. If $\beta_n = b_{r_n}$, then $\beta_n \to 0$ by (45). If $\beta_n > b_{r_n}$, then (100) holds; multiplying it by r_n, using compactness of I, and applying (106) gives

$$\beta_n - r_n s_n = r_n d_n \to 0.$$

Since $r_n s_n \to 0$, this also gives $\beta_n \to 0$. Thus $\beta_n \to 0$ in both cases. By (97), $y_n \to y$, and by the inverse relation together with (287) in Lemma A.1, $y(\beta_n) \to 1$. Hence

$$d_n \to y - 1 > 0.$$

Moreover, (100) in the interior case and (101) in the endpoint case give $\beta_n/r_n \geq s_n + d_n$. Since $d_n \to d := y - 1 > 0$ and $s_n \to \infty$, (105) is proved.

Case (c): $x = a(y)$. By (98), this is the only remaining subcase. Since $r_n s_n \to a(y)$ and $a(y_n) \to a(y)$, we have

$$r_n h_n = r_n s_n - a(y_n) \to a(y) - a(y) = 0.$$

The threshold equation $G_n(\tilde{a}) = 0$ is

$$g_{r_n}(\tilde{a}) = r_n(s_n + y_n) = a(y_n) + r_n y_n + r_n h_n = g_{r_n}\{a(y_n)\} + r_n h_n,$$

where $y\{a(y_n)\} = y_n$. Since $h_n < 0$ and g_{r_n} is strictly increasing by (59) and Lemma 2.14, the corresponding threshold lies below $a(y_n)$. To justify the existence and location of this threshold, choose $\eta > 0$ such that

$$[a(y_n) - \eta, a(y_n)] \subset (0, a_q)$$

for all large n; this is possible because $a(y_n) \to a(y) \in (0, a_q)$. Since $-r_n h_n \to 0$ and $g'_{r_n} \geq 1$, for all large n,

$$g_{r_n}\{a(y_n) - \eta\} < g_{r_n}\{a(y_n)\} + r_n h_n < g_{r_n}\{a(y_n)\}.$$

Continuity and strict monotonicity of g_{r_n} therefore give a unique $\tilde{\beta}_n \in (a(y_n) - \eta, a(y_n))$ satisfying

$$g_{r_n}(\tilde{\beta}_n) = g_{r_n}\{a(y_n)\} + r_n h_n.$$

Moreover, the mean-value theorem and $g'_{r_n} \geq 1$ give

$$0 < a(y_n) - \tilde{\beta}_n \leq -r_n h_n \longrightarrow 0.$$

Because $a(y_n)$ remains bounded away from 0 and a_q, and $b_{r_n} \to 0$ by (45) in Lemma 2.10, it follows that

$$b_{r_n} < \tilde{\beta}_n < a_q$$

eventually. Thus $\tilde{\beta}_n$ is the threshold β_n in (99), so $G_n(\beta_n) = 0$. The mean-value theorem applied to g_{r_n} now gives, for some ξ_n between β_n and $a(y_n)$,

$$-r_n h_n = g_{r_n}\{a(y_n)\} - g_{r_n}(\beta_n) = (a(y_n) - \beta_n)\{1 + r_n y'(\xi_n)\}.$$

Therefore $\beta_n < a(y_n)$ and

$$a(y_n) - \beta_n = \frac{-r_n h_n}{1 + r_n y'(\xi_n)}. \tag{107}$$

For some ζ_n between β_n and $a(y_n)$,

$$\frac{\beta_n}{r_n} - s_n = d_n = y'(\zeta_n)\{a(y_n) - \beta_n\}. \tag{108}$$

Combining (107) and (108) gives

$$s_n d_n = (r_n s_n)\frac{y'(\zeta_n)}{1 + r_n y'(\xi_n)}(-h_n). \tag{109}$$

Here $r_n s_n \to a(y) > 0$. Also $\beta_n \to a(y)$ and $a(y_n) \to a(y)$, so $\xi_n, \zeta_n \to a(y)$; hence $y'(\zeta_n) \to y'\{a(y)\} > 0$, while $1 + r_n y'(\xi_n) \to 1$. Since $h_n \to -\infty$ by (98), (109) yields

$$s_n d_n \longrightarrow \infty.$$

Thus (105) is proved.

This proves (93). Combining (80) and (93), and reducing γ if necessary, proves (88).

Proof of (89)**.** Apply Lemma 2.12 to $\bar{R}_{r,-y}$, with

$$m_r(y) = \frac{a(y)}{r}, \qquad w_r(y) = 1, \qquad d(y) = qM(y), \qquad \xi(y) = \kappa(y).$$

The set I is compact, and $a_I = \inf_{y \in I} a(y) > 0$; hence, for every fixed $H > 0$,

$$\inf_{y \in I} \left\{ \frac{a(y)}{r} - H \right\} \geq \frac{a_I}{r} - H > u_q$$

for all sufficiently small r. Because $I \subset (1, y_q)$ and M is continuous and strictly increasing there,

$$0 < d(y) = qM(y) < 1, \qquad \xi(y) = \kappa(y) > 0,$$

and both functions are continuous on I. Equation (87) is exactly (51), while (88) is (52). Thus all hypotheses of Lemma 2.12 hold, and (53) gives (89) uniformly on I.

Proof of (90)**.** Fix $H = 1$. For $y \in I$ and $|h| \leq H$, let $\tilde{\alpha}_r(y,h)$ be the threshold defined by (82). Write

$$\bar{R}_{2,r,-y}\left(\frac{a(y)}{r} + h\right) = \frac{qW_r\{\tilde{\alpha}_r(y,h)\}}{p(a(y)/r + h)} = [1 - qM\{y(\tilde{\alpha}_r(y,h))\}]\,\frac{p\{\tilde{\alpha}_r(y,h)/r\}}{p(a(y)/r + h)}.$$

Equation (79) shows that the primary non-null is not rejected for $y \in I$ and $|h| \leq H$. Consequently, (42) gives

$$\bar{R}_{r,-y}\left(\frac{a(y)}{r} + h\right) = \bar{R}_{0,r,-y}\left(\frac{a(y)}{r} + h\right) + \bar{R}_{2,r,-y}\left(\frac{a(y)}{r} + h\right).$$

We first differentiate the secondary non-null contribution $\bar{R}_{2,r,-y}\{a(y)/r + h\}$. Equation (60) holds uniformly, since

$$g_r(b_r) > 0 > ry - a(y) - rh = r\left\{ y - \left(\frac{a(y)}{r} + h\right) \right\}$$

for all sufficiently small r. Apply (64) in Lemma 2.14 with $c = a(y)/r + h$ and $\beta = \tilde{\alpha}_r(y,h)$. The term in (64),

$$-\frac{qM'\{y(\beta)\}y'(\beta)}{1 - qM\{y(\beta)\}}\,\frac{r}{1 + ry'(\beta)},$$

is $O(r)$ uniformly by (83) and (289) in Lemma A.1. Indeed, (83) places β, uniformly over $y \in I$ and $|h| \leq H$, in a fixed compact subinterval of $(0, a_q)$. On this subinterval,

$$\frac{|M'\{y(\beta)\}|y'(\beta)}{1 - qM\{y(\beta)\}} = O(1), \qquad \frac{r}{1 + ry'(\beta)} = O(r),$$

by (289) in Lemma A.1; hence their product is $O(r)$ uniformly.

The last two terms in (64) are

$$-\frac{\lambda(\beta/r)}{1+ry'(\beta)}+\lambda(c).$$

For these terms, (61) implies

$$\frac{\beta}{r}=c+y-y(\beta).$$

Because $a(y)$ is bounded away from zero on I, (83) gives $c \asymp \beta/r \asymp r^{-1}$ uniformly. Thus (293) in Lemma A.2 gives

$$\begin{aligned}\lambda(c)-\frac{\lambda(\beta/r)}{1+ry'(\beta)}&=c-\frac{\beta/r}{1+ry'(\beta)}+O(r)\\&=\frac{rcy'(\beta)+y(\beta)-y}{1+ry'(\beta)}+O(r).\end{aligned}$$

Now $rc=a(y)+rh\to a(y)$, while (83) and continuity of y and y' give

$$y(\beta)-y=o(1),\qquad y'(\beta)=y'\{a(y)\}+o(1)$$

uniformly. Consequently,

$$\lambda(c)-\frac{\lambda(\beta/r)}{1+ry'(\beta)}\longrightarrow a(y)y'\{a(y)\}=\kappa(y)$$

uniformly, by (77). Combining this with (91) yields

$$\frac{\partial}{\partial h}\bar{R}_{2,r,-y}\left(\frac{a(y)}{r}+h\right)\longrightarrow\kappa(y)\{1-qM(y)\}e^{\kappa(y)h} \tag{110}$$

uniformly for $y\in I$ and $|h|\le H$.

It remains to differentiate the null contribution. Use the notation of Lemma 2.14 at $c=a(y)/r+h$. Since $a(y)$ is bounded away from zero on I and $a(y)\le a_q$, for all sufficiently small r,

$$u_q\le\frac{a(y)}{r}+h\le\frac{a_q+1}{r}$$

uniformly for $y\in I$ and $|h|\le H$. Applying (47) in Lemma 2.11 with $A=a_q+1$ and $T=\sup_{u\in I}u$ shows that

$$\sup_{y\in I}\sup_{|h|\le H}\bar{R}_{0,r,-y}\left(\frac{a(y)}{r}+h\right)=O(1). \tag{111}$$

Equation (58) in Lemma 2.14 can be written as

$$\frac{\partial}{\partial h}\log\bar{R}_{0,r,-y}(c)=\sum_{\sigma\in\{-,+\}}\frac{Q_\sigma}{Q_++Q_-}\left\{\lambda(c)-\frac{\lambda(x_\sigma)}{\sqrt{1-r^2}}\right\},$$

because $\partial c/\partial h=1$. Moreover, (293) in Lemma A.2 gives, for either sign,

$$\lambda(c)-\frac{\lambda(x_\pm)}{\sqrt{1-r^2}}=c-\frac{c\pm ry}{1-r^2}+O(r)=O(r)$$

uniformly for $y\in I$ and $|h|\le H$. Since the nonnegative weights $Q_\sigma/(Q_++Q_-)$ sum to one,

$$\sup_{y\in I}\sup_{|h|\le H}\left|\frac{\partial}{\partial h}\log\bar{R}_{0,r,-y}\left(\frac{a(y)}{r}+h\right)\right|=O(r). \tag{112}$$

Multiplying (111) and (112) gives

$$\sup_{y\in I}\sup_{|h|\le H}\left|\frac{\partial}{\partial h}\bar{R}_{0,r,-y}\left(\frac{a(y)}{r}+h\right)\right|\longrightarrow 0. \tag{113}$$

Since I is compact and $\kappa(y)\{1-qM(y)\}e^{\kappa(y)h}$ is strictly positive on $I\times[-H,H]$, (110) and (113) imply

$$\inf_{y\in I}\inf_{|h|\le H}\frac{\partial}{\partial h}\bar{R}_{r,-y}\left(\frac{a(y)}{r}+h\right)>0 \tag{114}$$

for all sufficiently small r. For every fixed $H>0$, (92) verifies (54) with $d(y)=qM(y)$. The null contribution is smooth, and (114), with $H=1$, gives the positive derivative required by Lemma 2.13. That lemma therefore shows that $C_r(-y)$ is regular for every $y\in I$ when r is sufficiently small, and (55) gives

$$\sup_{y\in I}|D_r(-y)-qM(y)|\longrightarrow 0,$$

which is (90). □

2.6.4 The regime $z<-y_q$

Let $y=-z$. For $y>y_q$, let $\alpha_q(y)\in(0,a_q)$ be the unique solution of

$$qL_{\alpha_q(y)}(y)=1. \tag{115}$$

For fixed y, the map $a\mapsto L_a(y)$ is continuous and

$$\frac{\partial}{\partial a}\log L_a(y)=-a+y\tanh(ay). \tag{116}$$

By Lemma A.1, this derivative is positive on $(0,a(y))$. Since $y>y_q$, the same lemma gives $a(y)>a_q$, so $a\mapsto L_a(y)$ is strictly increasing on $[0,a_q]$. Moreover, by the definition of a_q in (36),

$$qL_0(y)=q<1,\quad qL_{a_q}(y)>qL_{a_q}(y_q)=qM(y_q)=1,$$

where the strict inequality follows from $a_q>0$, $y>y_q$, and the monotonicity of $\cosh(a_qy)$ in $y>0$. Thus continuity and strict monotonicity give exactly one solution to (115).

Lemma 2.20. *For every compact $I\subset(y_q,\infty)$, uniformly for $y\in I$,*

$$rC_r(-y)\longrightarrow\alpha_q(y) \tag{117}$$

and

$$D_r(-y)\longrightarrow 1. \tag{118}$$

Proof. **Proof of** (117)**.** Fix a compact $I\subset(y_q,\infty)$, and put

$$\delta_I=\inf_{y\in I}(y-y_q)>0.$$

Use $s_r(y)$ from (78) and $(\mathcal{A}_{r,y,s},\beta_{r,y,s})$ from (94). By (59) and Lemma 2.14, $g_r'(\tilde{a})=1+ry'(\tilde{a})>1$. For all sufficiently small r, the primary non-null is not rejected when $s>s_r(y)$. To verify the corresponding sign assertion for the secondary non-nulls, note that, conditional on $Z = -y$, the value associated with $\tilde{a} \in [b_r, a_q]$ is

$$\frac{\tilde{a}}{r} + y(\tilde{a}) - y \geq \frac{b_r}{r} + y(b_r) - \sup_{u \in I} u. \tag{119}$$

The monotonicity of $y(\cdot)$ gives (119). Its right-hand side tends to infinity as $r \downarrow 0$, by (45) in Lemma 2.10. Hence every secondary non-null value is positive for all sufficiently small r, uniformly over $y \in I$ and $\tilde{a} \in [b_r, a_q]$. Thus the rejected secondary non-nulls are indexed by $\mathcal{A}_{r,y,s}$. If this set is nonempty, continuity and strict monotonicity of g_r show that it is the closed interval $[\beta_{r,y,s}, a_q]$ and that

$$g_r(\beta_{r,y,s}) \geq r(s + y).$$

Consequently, in the nonempty case,

$$\frac{\beta_{r,y,s}}{r} - s \geq y - y(\beta_{r,y,s}) \geq y - y_q \geq \delta_I. \tag{120}$$

If $\mathcal{A}_{r,y,s} = \varnothing$, its μ_r-mass is zero. Since $\mu_r(\{a_q\}) = 0$ by (40), the convention in (94) gives in both cases

$$\mu_r(\mathcal{A}_{r,y,s}) = \mu_r([\beta_{r,y,s}, a_q]).$$

Using (40), (289) in Lemma A.1, and (120) when the set is nonempty, we obtain

$$0 \leq \frac{q\tau_r}{p(s)} \mu_r([\beta_{r,y,s}, a_q]) \leq \frac{p(s + \delta_I)}{p(s)}.$$

Moreover,

$$\inf_{y \in I} s_r(y) = c_r + y(b_r) - \sup_{y \in I} y \longrightarrow \infty,$$

because $c_r \to \infty$, $y(b_r) \to 1$, and I is compact. Therefore (303) in Lemma A.2 gives

$$\sup_{y \in I} \sup_{s > s_r(y)} \frac{q\tau_r}{p(s)} \mu_r([\beta_{r,y,s}, a_q]) \longrightarrow 0. \tag{121}$$

The function $(a, y) \mapsto L_a(y)$ is C^∞. Equation (116), together with $\alpha_q(y) < a_q < a(y)$ and Lemma A.1, shows that

$$\frac{\partial}{\partial a} \log L_a(y) \bigg|_{a = \alpha_q(y)} > 0.$$

The implicit function theorem therefore shows that $y \mapsto \alpha_q(y)$ is continuous on (y_q, ∞). Since I is compact,

$$m_I := \inf_{y \in I} \min\{\alpha_q(y), a_q - \alpha_q(y)\} > 0. \tag{122}$$

Fix any $0 < \delta < m_I/2$. For every $y \in I$, continuity and strict monotonicity in a give

$$qL_{\alpha_q(y) - \delta}(y) < qL_{\alpha_q(y)}(y) = 1 < qL_{\alpha_q(y) + \delta}(y).$$

Both strict gaps are continuous functions of y. Their minima over the compact set I are positive, so there is $\gamma > 0$ such that, uniformly on I,

$$qL_{\alpha_q(y) - \delta}(y) \leq 1 - 2\gamma, \quad qL_{\alpha_q(y) + \delta}(y) \geq 1 + 2\gamma \tag{123}$$

The choice $\delta < m_I/2$ also gives

$$\alpha_q(y) + \delta \le \alpha_q(y) + \frac{m_I}{2} \le \frac{\alpha_q(y) + a_q}{2} < a_q < a(y), \tag{124}$$

uniformly for $y \in I$.

Write $\psi_y(x) = \log L_x(y)$. The proof of Lemma A.1, through (290), shows that $\psi_y'(x) > 0$ for $0 < x < a(y)$. By (124), every $x \in (0, \alpha_q(y) + \delta)$ lies in $(0, a(y))$. Hence $\partial_x L_x(y) = L_x(y)\psi_y'(x) > 0$ throughout $(0, \alpha_q(y) + \delta)$, and therefore $x \mapsto L_x(y)$ is increasing on $[0, \alpha_q(y) + \delta]$.

On $u_q \le s \le s_r(y)$, (80) in Lemma 2.17 gives

$$\sup_{y \in I} \sup_{u_q \le s \le s_r(y)} \bar{R}_{r,-y}(s) \le q + o(1).$$

For $s > s_r(y)$, the term in (42) that corresponds to the primary non-nulls is zero, and (121) and (47) in Lemma 2.11 give, uniformly for $y \in I$ and $s_r(y) < s \le (\alpha_q(y) + \delta)/r$,

$$\bar{R}_{r,-y}(s) = \bar{R}_{0,r,-y}(s) + \frac{q\tau_r}{p(s)} \mu_r([\beta_{r,y,s}, a_q]) = qL_{rs}(y) + o(1).$$

Combining these two ranges gives

$$\sup_{u_q \le s \le (\alpha_q(y) - \delta)/r} \bar{R}_{r,-y}(s) \le qL_{\alpha_q(y)-\delta}(y) + o(1) \le 1 - \gamma. \tag{125}$$

Since $rs_r(y) \to 0$ uniformly on I and $\alpha_q(y) + \delta$ is bounded away from zero, we have

$$\frac{\alpha_q(y) + \delta}{r} > s_r(y)$$

for all sufficiently small r, uniformly on I. At the cutoff $(\alpha_q(y) + \delta)/r$, the term in (42) that corresponds to the primary non-nulls is zero, while the secondary term is nonnegative. Therefore (47) in Lemma 2.11 gives, uniformly for $y \in I$,

$$\bar{R}_{r,-y}\left(\frac{\alpha_q(y) + \delta}{r}\right) \ge \bar{R}_{0,r,-y}\left(\frac{\alpha_q(y) + \delta}{r}\right) = qL_{\alpha_q(y)+\delta}(y) + o(1). \tag{126}$$

By (123), the right side is at least $1 + \gamma$ for all sufficiently small r. Hence

$$\alpha_q(y) - \delta < rC_r(-y) \le \alpha_q(y) + \delta.$$

Letting $\delta \downarrow 0$ gives

$$rC_r(-y) \longrightarrow \alpha_q(y),$$

uniformly for $y \in I$.

Proof of (118)**.** We now verify regularity for each fixed sufficiently small r. Fix one $0 < \delta < m_I/2$ as in (122) and consider the interval

$$\frac{\alpha_q(y) - \delta}{r} \le s \le \frac{\alpha_q(y) + \delta}{r}.$$

The primary non-null has conditional value

$$T_i = c_r + y(b_r) - y = s_r(y).$$

It is positive for sufficiently small r, uniformly in $y \in I$, and it is rejected at cutoff s exactly when $s \le s_r(y)$. Since $\delta < m_I/2$ and $m_I \le \alpha_q(y)$, we have $\alpha_q(y) - \delta \ge m_I/2$. On the other hand,

$$rs_r(y) = rc_r + r\{y(b_r) - y\} \longrightarrow 0$$

uniformly for $y \in I$, by (46), $y(b_r) \to 1$, and compactness of I. Therefore, for all sufficiently small r, for every $y \in I$ and every cutoff s satisfying

$$\frac{\alpha_q(y) - \delta}{r} \le s \le \frac{\alpha_q(y) + \delta}{r},$$

we have

$$s_r(y) < \frac{\alpha_q(y) - \delta}{r} \le s.$$

Thus every cutoff in this interval lies strictly to the right of the primary value $s_r(y)$, so the primary non-null is not rejected there. Moreover, uniformly for these y and s,

$$\begin{aligned} r(s + y) - g_r(b_r) &\ge \alpha_q(y) - \delta + r\{y - y(b_r)\} - b_r \ge \frac{m_I}{2} - o(1) > 0, \\ g_r(a_q) - r(s + y) &\ge a_q - \alpha_q(y) - \delta + r(y_q - y) \ge \frac{m_I}{2} - o(1) > 0. \end{aligned}$$

Strict monotonicity of g_r therefore implies that $\mathcal{A}_{r,y,s} \ne \varnothing$, that $\beta_{r,y,s} \in (b_r, a_q)$. By (94), continuity and strict monotonicity of g_r give

$$g_r(\beta_{r,y,s}) = r(s + y).$$

Using $g_r(a) = a + ry(a)$ from (59), this equality yields

$$\beta_{r,y,s} = rs + r\{y - y(\beta_{r,y,s})\} = rs + O(r)$$

uniformly. Hence, for all sufficiently small r,

$$\frac{m_I}{4} \le \beta_{r,y,s} \le a_q - \frac{m_I}{4},$$

so the thresholds remain in a fixed compact subinterval of $(0, a_q)$.

For $y \in I$ and cutoffs s in the local crossing interval

$$\frac{\alpha_q(y) - \delta}{r} \le s \le \frac{\alpha_q(y) + \delta}{r},$$

write the secondary non-null contribution as

$$\bar{R}_{2,r,-y}(s) = \left[1 - qM\{y(\beta_{r,y,s})\}\right] \frac{p(\beta_{r,y,s}/r)}{p(s)}.$$

Then

$$\bar{R}_{r,-y}(s) = \bar{R}_{0,r,-y}(s) + \bar{R}_{2,r,-y}(s).$$

Abbreviate $\beta = \beta_{r,y,s}$. Equation (60) holds uniformly because $g_r(b_r) > 0 > r(y - s)$ for all sufficiently small r. The term in (64) of Lemma 2.14, evaluated at $c = s$ and $\beta_{r,y}(s) = \beta_{r,y,s} = \beta$, is

$$-\frac{qM'\{y(\beta)\}y'(\beta)}{1 - qM\{y(\beta)\}} \frac{r}{1 + ry'(\beta)}.$$

Because
$$\frac{m_I}{4} \le \beta \le a_q - \frac{m_I}{4},$$
(289) in Lemma A.1 and continuity give a constant $C_I < \infty$ such that
$$\sup_{m_I/4 \le a \le a_q - m_I/4} \left| \frac{qM'\{y(a)\}y'(a)}{1 - qM\{y(a)\}} \right| \le C_I.$$
Since $y'(\beta) > 0$,
$$0 < \frac{r}{1 + ry'(\beta)} \le r,$$
and therefore
$$\left| \frac{qM'\{y(\beta)\}y'(\beta)}{1 - qM\{y(\beta)\}} \frac{r}{1 + ry'(\beta)} \right| \le C_I r.$$

The expressions in (64) are
$$-\frac{\lambda(\beta/r)}{1 + ry'(\beta)} \qquad \text{and} \qquad \lambda(s).$$
To bound their sum, write
$$\lambda(t) = t + \varepsilon(t), \qquad 0 < \varepsilon(t) < t^{-1},$$
by (293) in Lemma A.2. On the cutoff range,
$$s \ge \frac{m_I}{2r}, \qquad \frac{\beta}{r} \ge \frac{m_I}{4r},$$
and hence $\varepsilon(s) = O(r)$ and $\varepsilon(\beta/r) = O(r)$ uniformly. Using $\beta/r = s + y - y(\beta)$, we obtain
$$\begin{aligned}\lambda(s) - \frac{\lambda(\beta/r)}{1 + ry'(\beta)} &= s - \frac{\beta/r}{1 + ry'(\beta)} + O(r) \\ &= \frac{rsy'(\beta) + y(\beta) - y}{1 + ry'(\beta)} + O(r) = O(1),\end{aligned}$$
uniformly, because rs, y, $y(\beta)$, and $y'(\beta)$ are uniformly bounded. Hence
$$\sup_{y \in I} \sup_{|rs - \alpha_q(y)| \le \delta} \left| \frac{\partial}{\partial s} \log \bar{R}_{2,r,-y}(s) \right| = O(1). \tag{127}$$

Moreover, $\beta_{r,y,s}/r - s \ge \delta_I$ by (120), so $\beta_{r,y,s}/r \ge s + \delta_I$. Since p is decreasing on $[0, \infty)$, and since (289) in Lemma A.1 gives $0 \le 1 - qM\{y(\beta_{r,y,s})\} \le 1$, we have
$$0 \le \bar{R}_{2,r,-y}(s) \le \frac{p(s + \delta_I)}{p(s)}.$$
Applying (300) in Lemma A.2 with $x = s$ and $\delta = \delta_I$ gives
$$0 \le \bar{R}_{2,r,-y}(s) \le \frac{p(s + \delta_I)}{p(s)} \le (1 + s^{-2}) \exp\left\{-s\delta_I - \frac{\delta_I^2}{2}\right\} = o(r), \tag{128}$$
uniformly on the interval. The last $o(r)$ follows because (122) and $0 < \delta < m_I/2$ imply
$$s \ge \frac{\alpha_q(y) - \delta}{r} \ge \frac{m_I}{2r}$$

whenever $y \in I$ and $|rs - \alpha_q(y)| \le \delta$, and therefore

$$r^{-1}(1+s^{-2}) \exp\left\{-s\delta_I - \frac{\delta_I^2}{2}\right\} \le r^{-1}(1+o(1)) \exp\left\{-\frac{m_I \delta_I}{2r} - \frac{\delta_I^2}{2}\right\} \longrightarrow 0$$

uniformly. Since $\partial_s \bar{R}_{2,r,-y}(s) = \bar{R}_{2,r,-y}(s) \partial_s \log \bar{R}_{2,r,-y}(s)$, (127) and (128) imply

$$\sup_{y \in I} \sup_{|rs - \alpha_q(y)| \le \delta} \left| \frac{\partial}{\partial s} \bar{R}_{2,r,-y}(s) \right| = o(r). \tag{129}$$

For the null contribution term, put

$$x_+ = \frac{s + ry}{\sqrt{1-r^2}}, \qquad x_- = \frac{s - ry}{\sqrt{1-r^2}}, \qquad Q_\pm = \frac{\overline{\Phi}(x_\pm)}{\overline{\Phi}(s)}.$$

Because $\alpha_q(y) - \delta \ge m_I/2$, the range in (127) implies

$$s \ge \frac{m_I}{2r} \longrightarrow \infty,$$

while rs and y range over compact sets. Therefore (48) in Lemma 2.11, with $z = -y$, gives uniformly

$$Q_+ = e^{-(rs)^2/2 - rsy} + o(1), \qquad Q_- = e^{-(rs)^2/2 + rsy} + o(1).$$

Indeed, $s/x_\pm = 1 + o(1)$, the $O(r^2)$ term in (48) is uniform, and its remainder tends to zero because $s \to \infty$ uniformly.

Next, (294) in Lemma A.2 gives

$$\begin{aligned} \frac{1}{r}\left\{\lambda(s) - \frac{\lambda(x_+)}{\sqrt{1-r^2}}\right\} &= -rs - y + o(1), \\ \frac{1}{r}\left\{\lambda(s) - \frac{\lambda(x_-)}{\sqrt{1-r^2}}\right\} &= -rs + y + o(1). \end{aligned} \tag{130}$$

More precisely, for either sign,

$$\lambda(s) - \frac{\lambda(x_\pm)}{\sqrt{1-r^2}} = s - \frac{s \pm ry}{1-r^2} + \frac{1}{s} - \frac{1}{s \pm ry} + O(r^3),$$

where

$$\frac{1}{s} - \frac{1}{s \pm ry} = O(r^3), \qquad \frac{1}{r}\left\{s - \frac{s \pm ry}{1-r^2}\right\} = \frac{-rs \mp y}{1-r^2}.$$

This proves (130).

Substitution of these four expansions into (57) in Lemma 2.14, together with $\pi_{0,r} \to 1$, yields

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial s}\bar{R}_{0,r,-y}(s) &= \frac{q}{2} e^{-(rs)^2/2} \left[e^{-rsy}(-rs - y) + e^{rsy}(-rs + y)\right] + o(1) \\ &= q e^{-(rs)^2/2}\{-rs \cosh(rsy) + y \sinh(rsy)\} + o(1). \end{aligned}$$

Finally, differentiating $L_a(y) = e^{-a^2/2}\cosh(ay)$ gives

$$\frac{\partial}{\partial a} L_a(y) = e^{-a^2/2}\{-a\cosh(ay) + y \sinh(ay)\}.$$

Evaluating at $a = rs$ proves

$$\frac{1}{r}\frac{\partial}{\partial s}\bar{R}_{0,r,-y}(s) = qe^{-(rs)^2/2}\{-rs\cosh(rsy) + y\sinh(rsy)\} + o(1) = q\frac{\partial}{\partial a}L_a(y)\Big|_{a=rs} + o(1). \quad (131)$$

By (122) and (124), the pairs (rs, y) under consideration range over a compact subset of $\{(a, y) : y \in I,\ 0 < a < a(y)\}$. Continuity, (116), and Lemma A.1 therefore show that $\partial_a L_a(y)|_{a=rs}$ is bounded away from zero uniformly. Combining (129) and (131) yields

$$\inf_{y\in I}\inf_{|rs-\alpha_q(y)|\le\delta}\frac{\partial}{\partial s}\bar{R}_{r,-y}(s) > 0 \quad (132)$$

for all sufficiently small r. Equation (117) gives, uniformly for $y \in I$,

$$\frac{\alpha_q(y) - \delta}{r} < C_r(-y) < \frac{\alpha_q(y) + \delta}{r}$$

for all sufficiently small r. Equation (132) and the smoothness of the null contribution for $|rs - \alpha_q(y)| \le \delta$ verify the hypotheses of Proposition 2.5; hence $C_r(-y)$ is regular.

Equation (44) gives

$$D_r(-y) = \bar{R}_{0,r,-y}\{C_r(-y)\}.$$

Because $rC_r(-y) \to \alpha_q(y)$ uniformly and $\alpha_q(y)$ stays in a compact subset of $(0, a_q)$, (47) in Lemma 2.11 applies at $c = C_r(-y)$ and gives

$$D_r(-y) = qL_{rC_r(-y)}(y) + o(1) \longrightarrow qL_{\alpha_q(y)}(y) = 1,$$

uniformly for $y \in I$, where the equality follows from (115). This proves (118). □

2.6.5 Conditional limit of D_r

The following lemma establishes the limit of $D_r(z)$ as $r \downarrow 0$, with D_* defined in Section 2.2.

Lemma 2.21. *For every $z \notin \{-1, -y_q\}$,*

$$D_r(z) \longrightarrow D_*(z). \quad (133)$$

Consequently,

$$\mathbb{E}[D_r(Z)] \longrightarrow \mathbb{E}[D_*(Z)] = \ell_=(q). \quad (134)$$

Proof. If $z > -1$, Lemma 2.15 gives $D_r(z) \to q$. If $-y_q < z < -1$, put $y = -z$ and apply Lemma 2.19. If $z < -y_q$, apply Lemma 2.20. This proves (133).

At every regular crossing, $0 \le D_r \le 1$; the value assigned at nonregular points also lies in $[0, 1]$. Dominated convergence gives the convergence in (134). Finally,

$$\begin{aligned}\mathbb{E}[D_*(Z)] &= q\mathbb{P}(Z > -1) + q\int_{-y_q}^{-1} M(-z)\phi(z)\,\mathrm{d}z + \mathbb{P}(Z < -y_q) \\ &= q\Phi(1) + q\int_1^{y_q} M(y)\phi(y)\,\mathrm{d}y + \overline{\Phi}(y_q) \\ &= \ell_=(q).\end{aligned}$$

□

2.7 Proofs of Theorem 2.1 and Propositions 2.2 and 2.3

Proof of Theorem 2.1. Lemma 2.21 gives $\mathbb{E}\{D_*(Z)\} = \ell_=(q)$. Fix $\delta > 0$. Choose a compact set

$$E \subset \mathbb{R} \setminus \{-1, -y_q\}$$

such that

$$\int_E D_*(z)\phi(z)\,\mathrm{d}z > \ell_=(q) - \frac{\delta}{4}.$$

Because E is compact and separated from -1 and $-y_q$, the sets

$$E_+ = E \cap (-1, \infty), \qquad E_0 = E \cap (-y_q, -1), \qquad E_- = E \cap (-\infty, -y_q)$$

are compact after discarding the empty ones. Apply the compact-uniform regime estimates on these pieces: (66)–(67) on E_+, (87)–(88) on E_0, and (125)–(126) on E_-, together with the corresponding null limits (47) and (92). Taking the minimum of the positive crossing margins over the nonempty pieces and then taking r sufficiently small, we obtain brackets

$$A_r(z) < B_r(z), \quad z \in E,$$

and a constant $\gamma > 0$, such that

$$\sup_{z\in E}\sup_{u_q \le c \le A_r(z)} \bar{R}_{r,z}(c) \le 1 - \gamma, \quad \inf_{z\in E} \bar{R}_{r,z}\{B_r(z)\} \ge 1 + \gamma,$$

$$\sup_{z\in E}\sup_{A_r(z) \le c \le B_r(z)} |\bar{R}_{0,r,z}(c) - D_*(z)| \le \frac{\delta}{8}.$$

Fix such an r and put

$$C_* = \sup_{z\in E} B_r(z) < \infty.$$

For $z \in E$, define

$$\mathcal{E}_N(z) = \max\left\{\|\widehat{R}_N - R_{r,z}\|_{\infty,[0,1]}, \|\widehat{R}_{0,N} - \pi_{0,r}R_{0,r,z}\|_{\infty,[0,1]}\right\}$$

and

$$t_* = \frac{p(C_*)}{q}\min\left\{\frac{\gamma}{2}, \frac{\delta}{16}\right\} > 0.$$

Following the argument in the proof of Theorem 2.8, on the event $\mathcal{E}_N(z) < t_*$, for $u_q \le c \le C_*$,

$$\left|\frac{q\widehat{R}_N\{p(c)\}}{p(c)} - \bar{R}_{r,z}(c)\right| \le \frac{q\mathcal{E}_N(z)}{p(C_*)},$$

$$\left|\frac{q\widehat{R}_{0,N}\{p(c)\}}{p(c)} - \bar{R}_{0,r,z}(c)\right| \le \frac{q\mathcal{E}_N(z)}{p(C_*)}.$$

The first bound and the two crossing margins imply

$$A_r(z) < \widehat{C}_N \le B_r(z).$$

The second bound, (25), and the null bracket then give

$$\mathrm{FDP} \ge \bar{R}_{0,r,z}(\widehat{C}_N) - \frac{\delta}{16} \ge D_*(z) - \frac{3\delta}{16}.$$

Lemma 2.6 gives, uniformly for $z \in E$,

$$\mathbb{P}\{\mathcal{E}_N(z) \geq t_* \mid Z = z\} \leq \eta_N := 4e^{-2Nt_*^2} \longrightarrow 0.$$

Since FDP ≥ 0 and $0 \leq D_* \leq 1$, conditional expectation and integration over E yield

$$\begin{aligned}\mathbb{E}_{\Omega_N, Z, \varepsilon}[\mathrm{FDP}] &\geq (1-\eta_N) \int_E \left(D_*(z) - \frac{3\delta}{16}\right)_+ \phi(z)\,\mathrm{d}z \\ &\geq \int_E D_*(z)\phi(z)\,\mathrm{d}z - \frac{3\delta}{16} - \eta_N.\end{aligned}$$

Choose N large enough that $\eta_N < \delta/16$. Using the choice of E and (29), we obtain

$$\mathbb{E}_{\Omega_N}[\mathrm{FDR}_{\Omega_N}(\mathrm{BH}_q)] > \ell_=(q) - \frac{\delta}{2}.$$

For such an N, some realization $\omega^\star$ satisfies

$$\mathrm{FDR}_{\omega^\star}(\mathrm{BH}_q) > \ell_=(q) - \frac{\delta}{2}.$$

By Lemma 2.9, choose $\eta > 0$ so small that

$$\mathrm{FDR}^{(\eta)}_{\omega^\star}(\mathrm{BH}_q) > \mathrm{FDR}_{\omega^\star}(\mathrm{BH}_q) - \frac{\delta}{2} > \ell_=(q) - \delta. \tag{135}$$

The perturbed vector $T^{(\eta)}$ has finite dimension N, mean vector $(\theta_i)_{i=1}^N \in \mathbb{R}^N$, and covariance matrix $\Sigma_{\omega^\star,\eta}$, which has unit diagonal and is strictly positive definite by (34) in Lemma 2.9. Thus the bound (135) establishes (6). Since $\delta > 0$ was arbitrary, taking the supremum proves (7). □

Proof of Proposition 2.2. Recall from Lemma A.1 that, for $y > 1$, $a(y)$ is the unique positive maximizer of $a \mapsto L_a(y)$, equivalently the unique positive solution of

$$a(y) = y \tanh\{a(y)y\}.$$

By the monotonicity of $a(\cdot)$ established in that lemma, $a(y) \geq a(2) > 0$ for $y \geq 2$, so $a(y)y \to \infty$. Moreover,

$$0 \leq y - a(y) = y\,[1 - \tanh\{a(y)y\}] \leq 2ye^{-2a(y)y} \longrightarrow 0.$$

Because $a(y)$ is the maximizer defining $M(y)$, we have $M(y) = L_{a(y)}(y)$. Substituting the definitions of $L_{a(y)}(y)$ and $\phi(y)$, expanding $\cosh(t) = (e^t + e^{-t})/2$, and completing the two squares gives

$$\begin{aligned}M(y)\phi(y) &= e^{-a(y)^2/2}\cosh\{a(y)y\}\frac{e^{-y^2/2}}{\sqrt{2\pi}} \\ &= \frac{1}{2\sqrt{2\pi}}\left[e^{-\{a(y)^2+y^2\}/2+a(y)y} + e^{-\{a(y)^2+y^2\}/2-a(y)y}\right] \\ &= \frac{1}{2\sqrt{2\pi}}\left[e^{-\{y-a(y)\}^2/2} + e^{-\{y+a(y)\}^2/2}\right].\end{aligned} \tag{136}$$

Consequently, $M(y)\phi(y) \to 1/(2\sqrt{2\pi})$. The difference $M(y)\phi(y) - 1/(2\sqrt{2\pi})$ is integrable on $[1,\infty)$. Indeed, for all sufficiently large y, $a(y) \geq y/2$, so

$$y - a(y) \leq 2ye^{-y^2},$$

while

$$\left| e^{-\{y-a(y)\}^2/2} - 1 \right| \le \frac{\{y-a(y)\}^2}{2}, \quad e^{-\{y+a(y)\}^2/2} \le e^{-y^2/2}.$$

It follows from the definition of c_ℓ that, as $Y \to \infty$,

$$\int_1^Y M(y)\phi(y)\,\mathrm{d}y = \frac{Y}{2\sqrt{2\pi}} + c_\ell - \Phi(1) + o(1). \tag{137}$$

Next, (136) gives

$$M(y) = \frac{1}{2}e^{y^2/2}\left[e^{-\{y-a(y)\}^2/2} + e^{-\{y+a(y)\}^2/2}\right] = \frac{1}{2}e^{y^2/2}\{1+o(1)\}.$$

Since $M(y_q) = 1/q \to \infty$, (287) in Lemma A.1 implies $y_q \to \infty$. The identity $qM(y_q) = 1$ therefore gives

$$y_q^2 = 2\log(1/q) + 2\log 2 + o(1), \quad y_q = \sqrt{2\log(1/q)} + o(1). \tag{138}$$

Equations (305) in Lemma A.2 and (138) give

$$\overline{\Phi}(y_q) \le \frac{\phi(y_q)}{y_q} = O\left(\frac{q}{y_q}\right) = o(q). \tag{139}$$

Substituting (137), (138), and (139) into (5) yields

$$\begin{aligned}
\ell_=(q) &= q\Phi(1) + q\left\{\frac{y_q}{2\sqrt{2\pi}} + c_\ell - \Phi(1) + o(1)\right\} + o(q) \\
&= \frac{q\sqrt{\log(1/q)}}{2\sqrt{\pi}} + c_\ell q + o(q).
\end{aligned}$$

Numerical quadrature of the absolutely convergent integral defining c_ℓ gives $c_\ell = 0.6492828\ldots$. ☐

Proof of Proposition 2.3. Since $q \in (0,1)$, (4) gives $M(y_q) = 1/q > 1$. By Lemma A.1, $M = 1$ on $[0,1]$ and M is strictly increasing on $(1,\infty)$, so $y_q > 1$ and $M(y) \ge 1$ for $1 \le y \le y_q$. Using (5),

$$\begin{aligned}
\ell_=(q) - q &= q\Phi(1) + q\int_1^{y_q} M(y)\phi(y)\,\mathrm{d}y + \overline{\Phi}(y_q) \\
&\quad - q\left\{\Phi(1) + \int_1^{y_q}\phi(y)\,\mathrm{d}y + \overline{\Phi}(y_q)\right\} \\
&= q\int_1^{y_q}\{M(y)-1\}\phi(y)\,\mathrm{d}y + (1-q)\overline{\Phi}(y_q).
\end{aligned}$$

The integral is nonnegative, and $(1-q)\overline{\Phi}(y_q) > 0$ because $q < 1$ and $y_q < \infty$. Hence $\ell_=(q) > q$. ☐

3 Lower bound for one-sided tests

3.1 Main result

For $0 < q < 1$, define

$$y_q^+ = \sqrt{2\log(1/q)} \tag{140}$$

and

$$\ell_\le(q) = \frac{q\sqrt{\log(1/q)}}{\sqrt{\pi}} + \frac{q}{2} + \overline{\Phi}(y_q^+). \tag{141}$$

The quantities in (140) and (141) determine the finite-dimensional one-sided Gaussian lower bound.

Theorem 3.1. *For every $q \in (0,1)$ and $\delta > 0$, there are $N < \infty$, $\theta \in \mathbb{R}^N$, and a positive definite correlation matrix Σ such that BH at level q, applied to the one-sided p-values $P_i = \overline{\Phi}(T_i)$ from $T \sim N(\theta, \Sigma)$, satisfies*

$$\mathrm{FDR}_{\theta,\Sigma}(\mathrm{BH}_q) > \ell_{\le}(q) - \delta.$$

Every true null in the construction has $\theta_i = 0$, and every non-null has $\theta_i > 0$.

The first proposition gives the small-q expansion of the one-sided lower bound.

Proposition 3.2. *As $q \downarrow 0$,*

$$\ell_{\le}(q) = \frac{q\sqrt{\log(1/q)}}{\sqrt{\pi}} + \frac{q}{2} + o(q).$$

In particular, $\ell_{\le}(q)/q \to \infty$.

The next proposition shows strict anticonservativeness at every nontrivial nominal level.

Proposition 3.3. *For every $q \in (0,1)$, $\ell_{\le}(q) > q$.*

Figure 2 in the Introduction plots $\ell_{\le}(q) - q$ and $\ell_{\le}(q)/q$ over the grid $q = 0.005k$, $k = 1, \ldots, 199$. Table 2 gives the absolute inflation $\ell_{\le}(q) - q$ and relative inflation $\ell_{\le}(q)/q$ for commonly used nominal levels.

3.2 Roadmap

Fix q. For a small $r > 0$, let $(\Theta_i, B_i)_{i=1}^N$ be i.i.d. from a law $\nu_{q,r}^+$, independently of $Z, \varepsilon_1, \ldots, \varepsilon_N$, which are independent standard normal variables, and set

$$T_i = \Theta_i + B_i Z + \sqrt{1 - B_i^2}\,\varepsilon_i. \tag{142}$$

In the model (142), the law $\nu_{q,r}^+$ has one null component and two non-null components. We first analyze the population BH crossing conditional on $Z = z$. For every $z \notin \{0, y_q^+\}$, its conditional FDP converges, as $r \downarrow 0$, to

$$D_+^*(z) = \begin{cases} q, & z \le 0, \\ q e^{z^2/2}, & 0 < z < y_q^+, \\ 1, & z \ge y_q^+. \end{cases} \tag{143}$$

The definition (143) specifies the values at $\{0, y_q^+\}$; these values do not affect the expectation because the boundary points have zero probability. Therefore

$$\mathbb{E}\{D_+^*(Z)\} = \frac{q}{2} + q \int_0^{y_q^+} e^{z^2/2} \phi(z)\, \mathrm{d}z + \overline{\Phi}(y_q^+) = \ell_{\le}(q).$$

We reuse the machinery and use similar notation in Subsections 2.3 and 2.4. Throughout this section, put $p_+(x) = \overline{\Phi}(x)$ and $u_s^+ = p_+^{-1}(s)$. For $x \ge 0$ and $0 < s \le 1/2$,

$$p_+(x) = \frac{p(x)}{2}, \qquad p_+^{-1}(s) = p^{-1}(2s). \tag{144}$$

Consequently, every upper-tail ratio, high-quantile expansion, and hazard bound for p in Lemma A.2 applies to p_+, with the factor $1/2$ canceling from tail ratios. Use the symbols

$$\pi_0^+,\ R_z^+,\ R_{0,z}^+,\ \bar{R}_z^+,\ \bar{R}_{0,z}^+,\ C^+(z),\ D^+(z)$$

as the one-sided analogues of the quantities in (8) and (11)–(16). In particular, the one-sided population cutoff is

$$C^+(z) = \inf\{c \ge u_q^+ : \bar{R}_z^+(c) \ge 1\},$$

and every one-sided use of Definition 2.4 has lower endpoint u_q^+ in place of u_q. The only change to (9)–(10) is

$$F_{\theta,b,z}^+(s) = \overline{\Phi}\left(\frac{u_s^+ - \theta - bz}{\sqrt{1-b^2}}\right), \qquad |b| < 1, \tag{145}$$

while $F_{\theta,b,z}^+(s) = \mathbf{1}\{\theta + bz \ge u_s^+\}$ for $|b| = 1$. We retain the empty-crossing convention $C^+(z) = \infty$, $D^+(z) = 0$, and the regularity definition in Definition 2.4.

The proofs of Lemmas 2.6 and 2.7 and Theorem 2.8 are p-value agnostic and apply after (145); the one-sided rejection set in Lemma 2.7 is $\{i : T_i \ge \widehat{C}_N\}$. We also use the localized DKW bracketing in the proof of Theorem 2.8. Finally, the perturbation (32)–(33) and Lemma 2.9 apply to the one-sided p-values as well, because their continuity argument does not use the two-sided form of p.

3.3 Construction of $\nu_{q,r}^+$

This is the one-sided, sign-reversed specialization of the three-component construction in Subsection 2.4: the signal offset $y(a)$ is replaced by a, the envelope term $M\{y(a)\}$ by $e^{a^2/2}$, and the non-null loading $+1$ by -1.

Take $r > 0$, and set

$$b_r^+ = r\, p_+^{-1}(r^2). \tag{146}$$

We henceforth assume r is sufficiently small that $r^2 < 1/2$ and $b_r^+ < y_q^+$. For $b_r^+ \le a \le y_q^+$, define

$$W_r^+(a) = \frac{\{1 - qe^{a^2/2}\}p_+(a/r)}{q}. \tag{147}$$

Both factors in the numerator are nonnegative and decreasing in a, and the first vanishes at $a = y_q^+$. Hence W_r^+ is continuous and decreasing. Put

$$\tau_r^+ = W_r^+(b_r^+), \qquad \mu_r^+([a, y_q^+]) = \frac{W_r^+(a)}{\tau_r^+}, \quad b_r^+ \le a \le y_q^+. \tag{148}$$

Thus μ_r^+ is a probability measure on $[b_r^+, y_q^+]$. Since $p_+(b_r^+/r) = r^2$, (144) and the quantile expansion (298) in Lemma A.2 gives $b_r^+ \to 0$, and hence

$$\tau_r^+ \sim \frac{1-q}{q} r^2. \tag{149}$$

Finally, set

$$\pi_{0,r}^+ = 1 - r - \tau_r^+, \qquad p_+(c_r^+) = \frac{q(r + \tau_r^+)}{1 - q\pi_{0,r}^+}. \tag{150}$$

For all sufficiently small r, these quantities are well-defined and $\pi_{0,r}^+ > 0$.

The law $\nu_{q,r}^+$ is displayed in Table 4. Equivalently, for bounded measurable f,

$$\int f \,\mathrm{d}\nu_{q,r}^+ = \pi_{0,r}^+ f(0, r) + r f(c_r^+ + b_r^+, -1) + \tau_r^+ \int_{b_r^+}^{y_q^+} f(a/r + a, -1)\, \mu_r^+(\,\mathrm{d}a). \tag{151}$$

Under (151), the nulls become more significant when Z is positive. Every non-null statistic is deterministic conditional on Z and decreases with Z. The construction is not PRDS. To see this

Table 4: One-sided population construction. The opposite loading signs are essential.

Type	Mass	Θ	B
Nulls	$\pi_{0,r}^{+}$	0	r
Primary non-nulls	r	$c_r^{+} + b_r^{+}$	-1
Secondary non-nulls	$\tau_r^{+}\mu_r^{+}(\,\mathrm{d}a)$	$a/r + a$	$-1,\ a \in [b_r^{+}, y_q^{+}]$

directly, pair a null i with a non-null j, and fix $v \in (0,1)$. Conditional on $P_i = s$, the event $\{P_j \geq v\}$, which is increasing in the p-values, has probability

$$\Phi\left(\frac{p_{+}^{-1}(v) - \theta_j + rp_{+}^{-1}(s)}{\sqrt{1-r^2}}\right),$$

strictly decreasing in s. This also exhibits the negative null-to-non-null covariance, opposite to the nonnegative covariance assumption that guarantees Gaussian one-sided PRDS [Benjamini and Yekutieli, 2001, Chi et al., 2025]. On the other hand, all null–null correlations equal r^2, so the null p-values are positively regression dependent and the full vector is PRDN in the terminology of Su [2018].

3.4 Population analysis

3.4.1 Preliminaries

The next lemma gives the one-sided counterparts of (45)–(46).

Lemma 3.4. *As $r \downarrow 0$,*

$$b_r^{+} \to 0, \qquad b_r^{+}/r \to \infty, \qquad c_r^{+} \to \infty, \qquad rc_r^{+} \to 0, \qquad b_r^{+}/r - c_r^{+} \to \infty. \tag{152}$$

Proof. By (146) and the quantile expansion (298) in Lemma A.2,

$$\frac{b_r^{+}}{r} = p_{+}^{-1}(r^2) \sim 2\sqrt{\log(1/r)} \to \infty, \qquad b_r^{+} \sim 2r\sqrt{\log(1/r)} \to 0.$$

Moreover, (149) gives $\tau_r^{+} = O(r^2)$ and hence $\pi_{0,r}^{+} \to 1$. Therefore (150) yields

$$p_{+}(c_r^{+}) \sim \frac{qr}{1-q}.$$

Another application of (298) in Lemma A.2 gives

$$c_r^{+} \sim \sqrt{2\log(1/r)}.$$

Thus $c_r^{+} \to \infty$ and $rc_r^{+} \to 0$, while

$$\frac{b_r^{+}}{r} - c_r^{+} \sim (2-\sqrt{2})\sqrt{\log(1/r)} \longrightarrow \infty.$$

□

Apply the reused population notation to $\nu^+_{q,r}$, and write the resulting quantities as $R^+_{r,z}$, $R^+_{0,r,z}$, $\bar R^+_{r,z}$, $\bar R^+_{0,r,z}$, $\bar R^+_{1,r,z}$, $\bar R^+_{2,r,z}$, $C^+_r(z)$, and $D^+_r(z)$. Then

$$\bar R^+_{r,z}(c) = \bar R^+_{0,r,z}(c) + \bar R^+_{1,r,z}(c) + \bar R^+_{2,r,z}(c), \tag{153}$$

$$\bar R^+_{0,r,z}(c) = q\pi^+_{0,r}\frac{p_+\{(c-rz)/\sqrt{1-r^2}\}}{p_+(c)}. \tag{154}$$

$$\bar R^+_{1,r,z}(c) = \frac{qr}{p_+(c)}\mathbf{1}\{c \le c^+_r + b^+_r - z\}, \tag{155}$$

$$\bar R^+_{2,r,z}(c) = \frac{q\tau^+_r}{p_+(c)}\mu^+_r\left(\left[\frac{r(c+z)}{1+r}, y^+_q\right]\right). \tag{156}$$

The next lemma derives the derivatives of the null and secondary non-null contributions.

Lemma 3.5. *For $c > 0$, put*

$$u = \frac{c-rz}{\sqrt{1-r^2}}.$$

The null contribution satisfies

$$\frac{\mathrm{d}}{\mathrm{d}c}\log \bar R^+_{0,r,z}(c) = \lambda(c) - \frac{\lambda(u)}{\sqrt{1-r^2}}, \tag{157}$$

$$\frac{\mathrm{d}}{\mathrm{d}c}\bar R^+_{0,r,z}(c) = \bar R^+_{0,r,z}(c)\left\{\lambda(c) - \frac{\lambda(u)}{\sqrt{1-r^2}}\right\}. \tag{158}$$

Define

$$\beta_{r,z}(c) = \frac{r(c+z)}{1+r}.$$

If $\beta_{r,z}(c) \in (b^+_r, y^+_q)$, then

$$\bar R^+_{2,r,z}(c) = \left[1 - qe^{\{\beta_{r,z}(c)\}^2/2}\right]\frac{p_+\{\beta_{r,z}(c)/r\}}{p_+(c)} \tag{159}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}c}\log \bar R^+_{2,r,z}(c) = -\frac{qr\beta_{r,z}(c)e^{\{\beta_{r,z}(c)\}^2/2}}{(1+r)\left[1 - qe^{\{\beta_{r,z}(c)\}^2/2}\right]} + \lambda(c) - \frac{\lambda\{\beta_{r,z}(c)/r\}}{1+r}. \tag{160}$$

Proof. Since

$$\frac{\mathrm{d}u}{\mathrm{d}c} = \frac{1}{\sqrt{1-r^2}}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\log p_+(t) = \frac{p'_+(t)}{p_+(t)} = -\lambda(t),$$

Equation (154) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}c}\log \bar R^+_{0,r,z}(c) &= \frac{\mathrm{d}}{\mathrm{d}c}\left[\log(q\pi^+_{0,r}) + \log p_+(u) - \log p_+(c)\right] \\ &= -\frac{\lambda(u)}{\sqrt{1-r^2}} + \lambda(c),\end{aligned}$$

which is (157). Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}c}\bar R^+_{0,r,z}(c) &= \bar R^+_{0,r,z}(c)\frac{\mathrm{d}}{\mathrm{d}c}\log \bar R^+_{0,r,z}(c) \\ &= \bar R^+_{0,r,z}(c)\left\{\lambda(c) - \frac{\lambda(u)}{\sqrt{1-r^2}}\right\},\end{aligned}$$

which proves (158).

Now suppose that $\beta_{r,z}(c) \in (b_r^+, y_q^+)$. By (156), (148), and (147),

$$\begin{aligned}\bar{R}^+_{2,r,z}(c) &= \frac{q\tau_r^+}{p_+(c)}\mu_r^+([\beta_{r,z}(c), y_q^+]) \\ &= \frac{qW_r^+\{\beta_{r,z}(c)\}}{p_+(c)} \\ &= \left[1 - qe^{\{\beta_{r,z}(c)\}^2/2}\right]\frac{p_+\{\beta_{r,z}(c)/r\}}{p_+(c)}.\end{aligned}$$

This proves (159). Moreover,

$$\frac{\partial\beta_{r,z}(c)}{\partial c} = \frac{r}{1+r}.$$

Since $\beta_{r,z}(c) < y_q^+$, we have $1 - qe^{\{\beta_{r,z}(c)\}^2/2} > 0$, and hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}c}\log\bar{R}^+_{2,r,z}(c) &= -\frac{q\beta_{r,z}(c)e^{\{\beta_{r,z}(c)\}^2/2}}{1 - qe^{\{\beta_{r,z}(c)\}^2/2}}\frac{\partial\beta_{r,z}(c)}{\partial c} \\ &\quad - \lambda\{\beta_{r,z}(c)/r\}\frac{1}{r}\frac{\partial\beta_{r,z}(c)}{\partial c} + \lambda(c) \\ &= -\frac{qr\beta_{r,z}(c)e^{\{\beta_{r,z}(c)\}^2/2}}{(1+r)[1 - qe^{\{\beta_{r,z}(c)\}^2/2}]} + \lambda(c) - \frac{\lambda\{\beta_{r,z}(c)/r\}}{1+r}.\end{aligned}$$

This proves (160). □

The next lemma gives the limit of the null contribution.

Lemma 3.6. *For every compact $X \subset [0,\infty)$ and $K \subset \mathbb{R}$, as $r \downarrow 0$ and $c \to \infty$, uniformly over $z \in K$ subject to $rc \in X$,*

$$\bar{R}^+_{0,r,z}(c) = q\exp\left\{(rc)z - \frac{(rc)^2}{2}\right\} + o(1). \tag{161}$$

In particular, if $c = c(r) \to \infty$ and $rc \to x \geq 0$, then the left-hand side converges to $q\exp(xz - x^2/2)$, uniformly for z in compact sets.

Proof. Set

$$\Delta_r = \frac{c - rz}{\sqrt{1-r^2}} - c.$$

Because rc and z range over compact sets while $c \to \infty$, uniformly over the stated range, $\Delta_r = O(r) = o(1)$ and

$$\begin{aligned}c\Delta_r &= c^2\left\{\frac{1}{\sqrt{1-r^2}} - 1\right\} - \frac{rzc}{\sqrt{1-r^2}} \\ &= \frac{1}{2}(rc)^2 - z(rc) + o(1),\end{aligned}$$

where the remainder is uniform. The hazard expansion (294) in Lemma A.2, together with $c \to \infty$ and $\Delta_r = o(1)$, gives

$$\log\frac{p_+(c+\Delta_r)}{p_+(c)} = -\int_c^{c+\Delta_r}\lambda(t)\ \mathrm{d}t = -c\Delta_r + o(1)$$

uniformly on the same range. Exponentiating, using $(c - rz)/\sqrt{1-r^2} = c + \Delta_r$, and then applying (154) together with $\pi^+_{0,r} \to 1$ proves (161). □

The following lemma is the one-sided analogue of Lemma 2.12.

Lemma 3.7. *Let K be compact. For $z \in K$, let $m_r(z) > u_q^+$ and $w_r(z) > 0$. Suppose that, for every fixed $H > 0$,*

$$\inf_{z\in K}\{m_r(z) - Hw_r(z)\} > u_q^+ \tag{162}$$

for all sufficiently small r, and

$$\sup_{z\in K}\sup_{|h|\le H}\left|\bar{R}^+_{r,z}\{m_r(z) + w_r(z)h\} - d(z) - \{1-d(z)\}e^{\xi(z)h}\right| \longrightarrow 0, \tag{163}$$

where $d : K \to [0,1)$ and $\xi : K \to (0,\infty)$ are continuous. Suppose also that, for every $h_0 > 0$,

$$\limsup_{r\downarrow 0}\sup_{z\in K}\sup_{u_q^+\le c\le m_r(z)-w_r(z)h_0}\bar{R}^+_{r,z}(c) < 1. \tag{164}$$

Then

$$\sup_{z\in K}\left|\frac{C_r^+(z) - m_r(z)}{w_r(z)}\right| \longrightarrow 0. \tag{165}$$

Proof. The proof of Lemma 2.12 applies with u_q, $\bar{R}_{r,z}$, and $C_r(z)$ replaced by u_q^+, $\bar{R}^+_{r,z}$, and $C_r^+(z)$, respectively. □

The following lemma is the one-sided analogue of Lemma 2.13.

Lemma 3.8. *Under the assumptions and notation of Lemma 3.7, assume in addition that, for every fixed $H > 0$,*

$$\sup_{z\in K}\sup_{|h|\le H}\left|\bar{R}^+_{0,r,z}\{m_r(z) + w_r(z)h\} - d(z)\right| \longrightarrow 0. \tag{166}$$

If, for some $H_* > 0$ and all sufficiently small r, the map

$$h \longmapsto \bar{R}^+_{r,z}\{m_r(z) + w_r(z)h\}$$

is differentiable with positive derivative on $(-H_*, H_*)$, and the map

$$h \longmapsto \bar{R}^+_{0,r,z}\{m_r(z) + w_r(z)h\}$$

is continuous there for every $z \in K$, then $C_r^+(z)$ is regular and

$$\sup_{z\in K}|D_r^+(z) - d(z)| \longrightarrow 0. \tag{167}$$

Proof. The proof of Lemma 2.13 applies with u_q, $\bar{R}_{r,z}$, $\bar{R}_{0,r,z}$, $C_r(z)$, and $D_r(z)$ replaced by u_q^+, $\bar{R}^+_{r,z}$, $\bar{R}^+_{0,r,z}$, $C_r^+(z)$, and $D_r^+(z)$, respectively. □

The next lemma shows that the primary non-null contribution is negligible uniformly away from $z = 0$.

Lemma 3.9. *For every compact $K \subset (0,\infty)$, uniformly for $z \in K$, the primary non-null contribution in* (153)*, whenever nonzero, is bounded by*

$$\frac{qr}{p_+(c_r^+ + b_r^+ - z)} = o(1). \tag{168}$$

Proof. Whenever the primary non-null contribution is nonzero, the indicator in (155) equals one, so $\bar{R}^+_{1,r,z}(c) = qr/p_+(c)$. Equation (155) also gives $c \le c_r^+ + b_r^+ - z$. Since p_+ is decreasing,

$$\frac{qr}{p_+(c)} \le \frac{qr}{p_+(c_r^+ + b_r^+ - z)}.$$

The right-hand side is the expression in (168). By (150), it can be written as

$$\frac{qr}{p_+(c_r^+ + b_r^+ - z)} = \frac{r(1 - q\pi^+_{0,r})}{r + \tau_r^+} \frac{p_+(c_r^+)}{p_+(c_r^+ + b_r^+ - z)}.$$

Fix a compact $K \subset (0, \infty)$, and put $\delta_{r,z} = z - b_r^+$. By (152) in Lemma 3.4, $\delta_{r,z}$ remains in a fixed compact subset of $(0, \infty)$, uniformly for $z \in K$, and $c_r^+ - \delta_{r,z} \to \infty$ uniformly. Therefore (303) in Lemma A.2, applied with $x = c_r^+ - \delta_{r,z}$, gives

$$\frac{p_+(c_r^+)}{p_+(c_r^+ + b_r^+ - z)} = \frac{p_+(x + \delta_{r,z})}{p_+(x)} \longrightarrow 0$$

uniformly for $z \in K$. Since $0 < r(1 - q\pi^+_{0,r})/(r + \tau_r^+) \le 1$, the expression in (168) is $o(1)$ uniformly. $\square$

The next lemma bounds the secondary non-null contribution in the two positive- z regimes.

Lemma 3.10. *For all $c, z \in \mathbb{R}$,*

$$0 \le \frac{q\tau_r^+}{p_+(c)} \mu_r^+ \left(\left[\frac{r(c+z)}{1+r}, y_q^+ \right] \right) \le \frac{p_+\{(c+z)/(1+r)\}}{p_+(c)}. \tag{169}$$

Proof. If $r(c+z)/(1+r) \le b_r^+$, then

$$q\tau_r^+ \le p_+(b_r^+/r) \le p_+\{(c+z)/(1+r)\}.$$

If $b_r^+ < r(c+z)/(1+r) < y_q^+$, then

$$\begin{aligned} q\tau_r^+ \mu_r^+ \left(\left[\frac{r(c+z)}{1+r}, y_q^+ \right] \right) &= \left[1 - q \exp\left\{ \frac{r^2(c+z)^2}{2(1+r)^2} \right\} \right] p_+\left(\frac{c+z}{1+r} \right) \\ &\le p_+\left(\frac{c+z}{1+r} \right). \end{aligned}$$

If $r(c+z)/(1+r) \ge y_q^+$, the contribution is zero. Dividing these bounds by $p_+(c)$ proves (169). $\square$

3.4.2 The regime $z < 0$

Lemma 3.11. *For every compact $K \subset (-\infty, 0)$, uniformly for $z \in K$,*

$$C_r^+(z) = c_r^+ + o\{(c_r^+)^{-1}\} \tag{170}$$

and

$$D_r^+(z) \longrightarrow q. \tag{171}$$

Moreover, for all sufficiently small r, $C_r^+(z)$ is regular for every $z \in K$.

Proof. **Proof of** (170)**.** Fix a compact $K \subset (-\infty, 0)$. Uniformly for $z \in K$ and bounded h, all primary and secondary non-nulls are rejected at the cutoff $c_r^+ + h/c_r^+$. Indeed, the primary statistic equals $c_r^+ + b_r^+ - z$, so its distance above the cutoff is $b_r^+ - z - h/c_r^+$. This converges to $-z > 0$ because $b_r^+ \to 0$, $c_r^+ \to \infty$, and h is bounded.

Among the secondary non-nulls, the smallest statistic is attained at $a = b_r^+$. Its distance above the cutoff is

$$\frac{b_r^+}{r} + b_r^+ - z - c_r^+ - \frac{h}{c_r^+} = \left(\frac{b_r^+}{r} - c_r^+\right) + b_r^+ - z - \frac{h}{c_r^+},$$

which tends to infinity by (152) in Lemma 3.4. Thus both gaps are positive for all sufficiently small r, uniformly when z ranges over a compact subset of $(-\infty, 0)$ and h remains bounded.

Since every non-null is rejected,

$$\bar{R}_{r,z}^+\left(c_r^+ + \frac{h}{c_r^+}\right) = \bar{R}_{0,r,z}^+\left(c_r^+ + \frac{h}{c_r^+}\right) + \frac{q(r + \tau_r^+)}{p_+(c_r^+ + h/c_r^+)}.$$

For every fixed $H > 0$, (152) in Lemma 3.4 gives, uniformly for $z \in K$ and $|h| \le H$,

$$c_r^+ + \frac{h}{c_r^+} \longrightarrow \infty, \qquad r\left(c_r^+ + \frac{h}{c_r^+}\right) \longrightarrow 0.$$

Therefore (161) in Lemma 3.6 directly yields

$$\bar{R}_{0,r,z}^+\left(c_r^+ + \frac{h}{c_r^+}\right) \longrightarrow q$$

uniformly for $z \in K$ and $|h| \le H$. For the combined non-null contribution, the calibration (150) gives the exact identity

$$\frac{q(r + \tau_r^+)}{p_+(c_r^+ + h/c_r^+)} = (1 - q\pi_{0,r}^+)\frac{p_+(c_r^+)}{p_+(c_r^+ + h/c_r^+)}. \tag{172}$$

The tail ratio in (172) tends to e^h by (302) in Lemma A.2, whereas $1 - q\pi_{0,r}^+ \to 1 - q$. Combining the two contributions gives

$$\bar{R}_{r,z}^+(c_r^+ + h/c_r^+) \longrightarrow q + (1-q)e^h, \qquad \bar{R}_{0,r,z}^+(c_r^+ + h/c_r^+) \longrightarrow q. \tag{173}$$

The first limit is below one for $h < 0$, equals one at zero, and is above one for $h > 0$.

We next exclude earlier crossings. By (152) in Lemma 3.4, uniformly for $z \in K$, all non-nulls are rejected when $u_q^+ \le c \le c_r^+$, and $(c - rz)/\sqrt{1 - r^2} > c$ for small r. Hence

$$\bar{R}_{r,z}^+(c) < q\pi_{0,r}^+ + \frac{q(r + \tau_r^+)}{p_+(c)} \le q\pi_{0,r}^+ + \frac{q(r + \tau_r^+)}{p_+(c_r^+)} = 1. \tag{174}$$

For every fixed $h_0 > 0$, (174) and the calibration (150) give, uniformly for $z \in K$,

$$\begin{aligned} \sup_{u_q^+ \le c \le c_r^+ - h_0/c_r^+} \bar{R}_{r,z}^+(c) &\le q\pi_{0,r}^+ + (1 - q\pi_{0,r}^+)\frac{p_+(c_r^+)}{p_+(c_r^+ - h_0/c_r^+)} \\ &\longrightarrow q + (1-q)e^{-h_0} < 1. \end{aligned} \tag{175}$$

Thus Lemma 3.7 applies with

$$m_r(z) = c_r^+, \qquad w_r(z) = \frac{1}{c_r^+}, \qquad d(z) = q, \qquad \xi(z) = 1.$$

Equations (173) and (175) verify (163) and (164), respectively. Equation (165) therefore gives (170).

Proof of (171)**.** To apply Lemma 3.8, it remains to prove that the derivative of $\bar{R}^{+}_{r,z}(c^{+}_r + h/c^{+}_r)$ with respect to h is positive there. Fix $H < \infty$. Uniformly for $z \in K$ and $|h| \le H$, put

$$c = c^{+}_r + \frac{h}{c^{+}_r}, \qquad u = \frac{c - rz}{\sqrt{1-r^2}}.$$

Equation (152) in Lemma 3.4 gives $c/c^{+}_r \to 1$ and $u/c^{+}_r \to 1$ uniformly. Hence (293) in Lemma A.2 gives

$$\begin{aligned}\lambda(c) - \frac{\lambda(u)}{\sqrt{1-r^2}} &= c - \frac{u}{\sqrt{1-r^2}} + O\{(c^{+}_r)^{-1}\}\\ &= \frac{rz - r^2 c}{1-r^2} + O\{(c^{+}_r)^{-1}\}\\ &= O_{K,H}\left(r + r^2 c^{+}_r + \frac{1}{c^{+}_r}\right) = o(c^{+}_r).\end{aligned}$$

Thus the hazard difference in (158) is $o(c^{+}_r)$ uniformly. Applying (158) in Lemma 3.5, differentiating the fully rejected non-null contribution directly using $p'_{+}(c) = -\lambda(c)p_{+}(c)$, and then using (173), the calibration (150), and (302) in Lemma A.2, gives

$$\frac{\partial}{\partial h}\bar{R}^{+}_{0,r,z}\left(c^{+}_r + \frac{h}{c^{+}_r}\right) \longrightarrow 0, \qquad \frac{\partial}{\partial h}\frac{q(r+\tau^{+}_r)}{p_{+}(c^{+}_r + h/c^{+}_r)} \longrightarrow (1-q)e^{h},$$

$$\frac{\partial}{\partial h}\bar{R}^{+}_{r,z}\left(c^{+}_r + \frac{h}{c^{+}_r}\right) \longrightarrow (1-q)e^{h}.$$

Since $\inf_{|h|\le H}(1-q)e^{h} = (1-q)e^{-H} > 0$, for all sufficiently small r,

$$\inf_{z\in K}\inf_{|h|\le H}\frac{\partial}{\partial h}\bar{R}^{+}_{r,z}\left(c^{+}_r + \frac{h}{c^{+}_r}\right) > 0. \tag{176}$$

The second limit in (173) verifies (166). The derivative bound (176) and continuity of the null contribution verify the remaining hypotheses of Lemma 3.8. That lemma therefore gives regularity, and (167) gives (171), uniformly on K. ☐

3.4.3 The regime $z \in (0, y^{+}_q)$

Lemma 3.12. *For every compact $K \subset (0, y^{+}_q)$, uniformly for $z \in K$,*

$$C^{+}_r(z) = \frac{z}{r} + o(1) \tag{177}$$

and

$$D^{+}_r(z) \longrightarrow q e^{z^2/2}. \tag{178}$$

Moreover, for all sufficiently small r, $C^{+}_r(z)$ is regular for every $z \in K$.

Proof. **Proof of** (177)**.** Fix a compact $K \subset (0, y^{+}_q)$ and $H > 0$, and put $c = z/r + h$. On $z \in K$ and $|h| \le H$, the primary non-null contribution in (153) is nonzero only if

$$c^{+}_r + b^{+}_r - z \ge \frac{z}{r} + h.$$

Let $\delta_K = \inf_{z\in K} z > 0$. Uniformly for $z \in K$ and $|h| \le H$,

$$
\begin{aligned}
r\left\{\frac{z}{r} + h - (c_r^+ + b_r^+ - z)\right\} &= z + rh - rc_r^+ - rb_r^+ + rz \\
&\ge \delta_K - rH - rc_r^+ - rb_r^+ + r\delta_K \longrightarrow \delta_K > 0
\end{aligned}
$$

by (152) in Lemma 3.4. Hence

$$
c_r^+ + b_r^+ - z < \frac{z}{r} + h \tag{179}
$$

for all sufficiently small r, uniformly for $z \in K$ and $|h| \le H$, so the primary non-null contribution is absent.

A secondary non-null is rejected if and only if

$$
a \ge \beta_{r,z}(c) = z + \frac{rh}{1+r}.
$$

Compactness of $K \subset (0, y_q^+)$ implies that $\beta_{r,z}(c) \in (b_r^+, y_q^+)$ for all sufficiently small r, uniformly for $z \in K$ and $|h| \le H$. In the formula for $\bar{R}^+_{2,r,z}(c)$ in Lemma 3.5,

$$
\beta_{r,z}(c) \longrightarrow z, \qquad \frac{\beta_{r,z}(c)}{r} = \frac{z}{r} + \frac{h}{1+r},
$$

uniformly on $K \times [-H, H]$, so its first factor converges to $1 - qe^{z^2/2}$. With $x = z/r$, (297) in Lemma A.2 and the identity $p_+(x+u)/p_+(x+v) = p(x+u)/p(x+v)$ give

$$
\begin{aligned}
\log \frac{p_+\{x + h/(1+r)\}}{p_+(x+h)} &= \log \frac{x+h}{x + h/(1+r)} - x\left\{\frac{h}{1+r} - h\right\} - \frac{1}{2}\left\{\frac{h^2}{(1+r)^2} - h^2\right\} + o(1) \\
&= \log\left\{1 + \frac{hr}{(1+r)x + h}\right\} + \frac{zh}{1+r} + \frac{h^2 r(2+r)}{2(1+r)^2} + o(1) \\
&= zh + o(1) \qquad \text{uniformly on } K \times [-H, H].
\end{aligned}
$$

Consequently,

$$
\bar{R}^+_{2,r,z}(z/r + h) \longrightarrow \{1 - qe^{z^2/2}\}e^{zh}. \tag{180}
$$

For $c = z/r + h$, uniformly for $z \in K$ and $|h| \le H$, we have $c \to \infty$ and $rc = z + rh$. By uniformity of (161) in Lemma 3.6,

$$
\sup_{z\in K} \sup_{|h|\le H} \left|\bar{R}^+_{0,r,z}(z/r + h) - qe^{z^2/2}\right| \longrightarrow 0.
$$

Combining this with (180) and the absence of the primary non-null contribution for $z \in K$ and $|h| \le H$ gives

$$
\bar{R}^+_{r,z}(z/r + h) \longrightarrow qe^{z^2/2} + \{1 - qe^{z^2/2}\}e^{zh}, \qquad \bar{R}^+_{0,r,z}(z/r + h) \longrightarrow qe^{z^2/2}. \tag{181}
$$

Because $qe^{z^2/2} < 1$ uniformly away from one on K, the first limit crosses one strictly at $h = 0$.

We now verify (164) in Lemma 3.7. For every fixed $\eta > 0$, we show that

$$
\limsup_{r\downarrow 0} \sup_{z\in K} \sup_{u_q^+ \le c \le z/r - \eta} \bar{R}^+_{r,z}(c) < 1. \tag{182}
$$

Suppose instead that this assertion failed. Then there would be sequences satisfying

$$
r_n \downarrow 0, \qquad z_n \in K, \qquad u_q^+ \le c_n \le \frac{z_n}{r_n} - \eta, \qquad \bar{R}^+_{r_n,z_n}(c_n) \ge 1 - o(1).
$$

After passing to subsequences,

$$z_n \longrightarrow z_0, \qquad r_n c_n \longrightarrow x \in [0, z_0].$$

We consider three cases to get a contradiction.

Case (a). If $c_n = O(1)$, then, since $z_n \in K$,

$$\Delta_n := \frac{c_n - r_n z_n}{\sqrt{1 - r_n^2}} - c_n = c_n \left\{ \frac{1}{\sqrt{1 - r_n^2}} - 1 \right\} - \frac{r_n z_n}{\sqrt{1 - r_n^2}} = o(1).$$

Moreover, $\inf_n p_+(c_n) > 0$. Hence continuity of p_+, $\pi^+_{0,r_n} \to 1$, and (154) give

$$\frac{p_+(c_n + \Delta_n)}{p_+(c_n)} = 1 + o(1), \qquad \bar{R}^+_{0,r_n,z_n}(c_n) = q\pi^+_{0,r_n} \frac{p_+(c_n + \Delta_n)}{p_+(c_n)} = q + o(1).$$

By (153),

$$\begin{aligned}
0 &\le \bar{R}^+_{r_n,z_n}(c_n) - \bar{R}^+_{0,r_n,z_n}(c_n) \\
&= \frac{q r_n}{p_+(c_n)} \mathbf{1}\{c^+_{r_n} + b^+_{r_n} - z_n \ge c_n\} + \frac{q\tau^+_{r_n}}{p_+(c_n)} \mu^+_{r_n}\left(\left[\frac{r_n(c_n + z_n)}{1 + r_n}, y^+_q\right]\right) \\
&\le \frac{q(r_n + \tau^+_{r_n})}{p_+(c_n)} = O(r_n) = o(1),
\end{aligned}$$

The equality $q(r_n + \tau^+_{r_n})/p_+(c_n) = O(r_n)$ uses (149). Thus $\bar{R}^+_{r_n,z_n}(c_n) = q + o(1) < 1$, a contradiction.

Case (b). $c_n \to \infty$ and $x < z_0$. Put $d_n = (c_n + z_n)/(1 + r_n)$. Then

$$d_n - c_n = \frac{z_n - r_n c_n}{1 + r_n} \longrightarrow z_0 - x > 0, \qquad c_n(d_n - c_n) \longrightarrow \infty.$$

By (169) in Lemma 3.10 and (293) in Lemma A.2, the secondary non-null contribution satisfies

$$0 \le \frac{q\tau^+_{r_n}}{p_+(c_n)} \mu^+_{r_n}\left(\left[\frac{r_n(c_n + z_n)}{1 + r_n}, y^+_q\right]\right) \le \frac{p_+(d_n)}{p_+(c_n)} = \exp\left\{-\int_{c_n}^{d_n} \lambda(t)\, dt\right\} \le e^{-c_n(d_n - c_n)} = o(1).$$

Equations (168) and (161) in Lemmas 3.9 and 3.6 therefore yield

$$\bar{R}^+_{r_n,z_n}(c_n) = q \exp\left(x z_0 - \frac{x^2}{2}\right) + o(1) \le q e^{z_0^2/2} + o(1) < 1,$$

again a contradiction.

Case (c). $c_n \to \infty$ and $x = z_0$. Write $h_n = c_n - z_n/r_n \le -\eta$. If h_n is bounded, (181) gives a limit strictly below one. If $h_n \to -\infty$, then

$$d_n - c_n = -\frac{r_n h_n}{1 + r_n} > 0, \qquad c_n(d_n - c_n) \sim -z_0 h_n \longrightarrow \infty.$$

The hazard bound (293) in Lemma A.2 gives

$$\frac{p_+(d_n)}{p_+(c_n)} = \exp\left\{-\int_{c_n}^{d_n} \lambda(t)\, dt\right\} \le \exp\{-c_n(d_n - c_n)\} \longrightarrow 0.$$

Hence (169) makes the secondary non-null contribution $o(1)$, while (168) makes the primary non-null contribution $o(1)$. Since $r_n c_n \to x = z_0$ and $z_n \to z_0$, (161) gives

$$\bar{R}^+_{0,r_n,z_n}(c_n) \longrightarrow q \exp\left(z_0^2 - \frac{z_0^2}{2}\right) = qe^{z_0^2/2} < 1.$$

Therefore

$$\bar{R}^+_{r_n,z_n}(c_n) = \bar{R}^+_{0,r_n,z_n}(c_n) + o(1) \longrightarrow qe^{z_0^2/2} < 1,$$

again a contradiction.

Since all three cases lead to a contradiction, (182) holds. Apply Lemma 3.7 with

$$m_r(z) = \frac{z}{r}, \qquad w_r(z) = 1, \qquad d(z) = qe^{z^2/2}, \qquad \xi(z) = z.$$

Equations (181) and (182) verify (163) and (164), respectively. Therefore (165) yields (177).

Proof of (178)**.** We apply Lemma 3.8. Fix $H > 0$, put

$$c = \frac{z}{r} + h, \qquad \beta_{r,z}(c) = z + \frac{rh}{1+r}, \qquad |h| \le H.$$

Equation (179) shows that the primary non-null contribution is absent uniformly on this set. Put $u = (c - rz)/\sqrt{1-r^2}$. The hazard expansion (294) in Lemma A.2 gives

$$\lambda(c) - \frac{\lambda(u)}{\sqrt{1-r^2}} = o(1)$$

uniformly for $z \in K$ and $|h| \le H$. In the secondary log derivative (160), the first term is $O(r)$, while the same hazard expansion gives

$$\lambda(c) - \frac{\lambda\{\beta_{r,z}(c)/r\}}{1+r} = z + o(1)$$

uniformly. Combining these two estimates with (180) yields

$$\sup_{z\in K} \sup_{|h|\le H} \left| \frac{\partial}{\partial h} \bar{R}^+_{r,z}(z/r + h) - z\{1 - qe^{z^2/2}\}e^{zh} \right| \longrightarrow 0.$$

Since

$$\inf_{z\in K} \inf_{|h|\le H} z\{1 - qe^{z^2/2}\}e^{zh} > 0,$$

the derivative is positive on every fixed local window for all sufficiently small r. Equation (177) verifies (165), and the second limit in (181) verifies (166). Continuity follows from (154). Lemma 3.8 therefore gives regularity, and (167) gives (178), uniformly on K. □

3.4.4 The regime $z > y_q^+$

For $z > y_q^+$, define

$$\alpha_q^+(z) = z - \sqrt{z^2 - (y_q^+)^2} \in (0, z).$$

Lemma 3.13. *For every compact $K \subset (y_q^+, \infty)$, uniformly for $z \in K$,*

$$rC_r^+(z) \longrightarrow \alpha_q^+(z) \tag{183}$$

and

$$D_r^+(z) \longrightarrow 1. \tag{184}$$

Moreover, for all sufficiently small r, $C_r^+(z)$ is regular for every $z \in K$.

Proof. **Proof of** (183)**.** Fix a compact $K \subset (y_q^+, \infty)$. Fix $z \in K$ and $x \in (0, z)$. The limits below also hold uniformly when (x, z) ranges over a compact subset of $\{(x, z) : z \in K,\ 0 < x < z\}$. Then (152) in Lemma 3.4 implies that the primary non-null is not rejected at $c = x/r$. For the secondary non-null contribution, put $d = (x/r + z)/(1 + r)$. Then

$$d - \frac{x}{r} = \frac{z - x}{1 + r} > 0,$$

and (169) together with (293) in Lemma A.2 gives

$$0 \le \bar{R}_{2,r,z}^+(x/r) \le \frac{p_+(d)}{p_+(x/r)} \le \exp\{-(x/r)(d - x/r)\} \longrightarrow 0.$$

Equation (161) in Lemma 3.6, applied with $c = x/r$, gives

$$\bar{R}_{r,z}^+(x/r) \longrightarrow q \exp\left(xz - \frac{x^2}{2}\right), \tag{185}$$

with the same limit for the null contribution. The equation

$$q \exp\left(xz - \frac{x^2}{2}\right) = 1$$

has smaller positive solution $\alpha_q^+(z)$. The limit in (185) is below one for $x < \alpha_q^+(z)$ and above one for $\alpha_q^+(z) < x < z$.

We now establish the uniform gap needed to exclude earlier crossings. Fix $\eta > 0$ uniformly small enough that

$$0 < \alpha_q^+(z) - \eta < \alpha_q^+(z) + \eta < z, \qquad z \in K.$$

We claim that

$$\limsup_{r \downarrow 0} \sup_{z \in K} \sup_{u_q^+ \le c \le \{\alpha_q^+(z) - \eta\}/r} \bar{R}_{r,z}^+(c) < 1. \tag{186}$$

If (186) failed, there would be sequences

$$r_n \downarrow 0, \qquad z_n \in K, \qquad u_q^+ \le c_n \le \frac{\alpha_q^+(z_n) - \eta}{r_n}, \qquad \bar{R}_{r_n,z_n}^+(c_n) \ge 1 - o(1).$$

After taking a subsequence,

$$z_n \longrightarrow z_0 \in K, \qquad r_n c_n \longrightarrow x \in [0, \alpha_q^+(z_0) - \eta].$$

If (c_n) is bounded, then $p_+(c_n)$ is bounded away from zero, (154) gives $\bar{R}_{0,r_n,z_n}^+(c_n) = q + o(1)$, and (153) and (149) give

$$0 \le \bar{R}_{r_n,z_n}^+(c_n) - \bar{R}_{0,r_n,z_n}^+(c_n) \le \frac{q(r_n + \tau_{r_n}^+)}{p_+(c_n)} = o(1).$$

Thus $\bar{R}^+_{r_n,z_n}(c_n) \to q < 1$, a contradiction.

Suppose instead that $c_n \to \infty$, and define $d_n = (c_n + z_n)/(1 + r_n)$. Since

$$d_n - c_n = \frac{z_n - r_n c_n}{1 + r_n} \longrightarrow z_0 - x \geq z_0 - \alpha_q^+(z_0) + \eta > 0,$$

(169) in Lemma 3.10 and (293) in Lemma A.2 give

$$0 \leq \bar{R}^+_{2,r_n,z_n}(c_n) \leq \exp\{-c_n(d_n - c_n)\} = o(1).$$

Equation (168) in Lemma 3.9 gives $\bar{R}^+_{1,r_n,z_n}(c_n) = o(1)$, and (161) in Lemma 3.6 gives

$$\bar{R}^+_{r_n,z_n}(c_n) \longrightarrow q \exp\left(x z_0 - \frac{x^2}{2}\right) \leq q \exp\left(\{\alpha_q^+(z_0) - \eta\} z_0 - \frac{\{\alpha_q^+(z_0) - \eta\}^2}{2}\right) < 1,$$

where the inequality uses $x \leq \alpha_q^+(z_0) - \eta < z_0$. This is again a contradiction, so (186) holds. In particular, no crossing occurs at or before $\{\alpha_q^+(z) - \eta\}/r$. At $c = \{\alpha_q^+(z) + \eta\}/r$, the null contribution alone is larger than one for all sufficiently small r, uniformly on K. Therefore

$$\alpha_q^+(z) - \eta < rC_r^+(z) < \alpha_q^+(z) + \eta.$$

Letting $\eta \downarrow 0$ proves

$$rC_r^+(z) \longrightarrow \alpha_q^+(z)$$

uniformly on K.

Proof of (184)**.** It remains to prove regularity. Choose $\eta > 0$ such that, for every $z \in K$,

$$0 < \alpha_q^+(z) - 2\eta < \alpha_q^+(z) + 2\eta < \min\{z, y_q^+\}.$$

For any $x \in [\alpha_q^+(z) - \eta, \alpha_q^+(z) + \eta]$, uniformly over $z \in K$,

$$r\{x/r - (c_r^+ + b_r^+ - z)\} = x - rc_r^+ - rb_r^+ + rz > 0$$

by (152) in Lemma 3.4, so the primary non-null contribution is zero. Applying (294) in Lemma A.2 directly to (157) in Lemma 3.5 gives, uniformly for $z \in K$ and $x \in [\alpha_q^+(z) - \eta, \alpha_q^+(z) + \eta]$,

$$\frac{\mathrm{d}}{\mathrm{d}c} \log \bar{R}^+_{0,r,z}(c)\bigg|_{c=x/r} = r(z - x) + o(r). \tag{187}$$

Set

$$c_1 = \inf_{z\in K} \alpha_q^+(z) - \eta > 0, \qquad \delta = \frac{1}{2}\left\{\inf_{z\in K}[z - \alpha_q^+(z)] - \eta\right\} > 0.$$

The null limit (161) in Lemma 3.6 and (187) imply that, for some $c_0 > 0$,

$$\begin{aligned} \inf_{z\in K} \inf_{x\in[\alpha_q^+(z)-\eta,\alpha_q^+(z)+\eta]} \bar{R}^+_{0,r,z}(x/r) &\geq c_0, \\ \inf_{z\in K} \inf_{x\in[\alpha_q^+(z)-\eta,\alpha_q^+(z)+\eta]} \frac{\mathrm{d}}{\mathrm{d}c}\bar{R}^+_{0,r,z}(c)\bigg|_{c=x/r} &\geq c_0 \delta r. \end{aligned} \tag{188}$$

for all sufficiently small r. For the secondary non-null contribution, put $d = (x/r + z)/(1 + r)$. Here $rd \in (b_r^+, y_q^+)$, $x/r \geq c_1/r$, and $d - x/r \geq \delta$. Thus (160) in Lemma 3.5 and (293) in Lemma A.2 yield

$$0 \leq \bar{R}^+_{2,r,z}(x/r) \leq \frac{p_+(d)}{p_+(x/r)} \leq e^{-(x/r)(d-x/r)} \leq e^{-c_1\delta/r},$$

$$\left| \frac{\mathrm{d}}{\mathrm{d}c} \bar{R}^+_{2,r,z}(c) \Big|_{c=x/r} \right| = O(r^{-1} e^{-c_1\delta/r}) = o(r).$$

Combining this bound with (188),

$$\inf_{z \in K} \inf_{x \in [\alpha_q^+(z) - \eta, \alpha_q^+(z) + \eta]} \frac{\mathrm{d}}{\mathrm{d}c} \bar{R}^+_{r,z}(c) \Big|_{c=x/r} > 0.$$

Equation (183) gives

$$\frac{\alpha_q^+(z) - \eta}{r} < C_r^+(z) < \frac{\alpha_q^+(z) + \eta}{r}$$

for all sufficiently small r, uniformly for $z \in K$. Hence Proposition 2.5 proves regularity uniformly on K.

At the population cutoff, regularity gives $\bar{R}^+_{r,z}\{C_r^+(z)\} = 1$. By (183), uniformly for $z \in K$,

$$rC_r^+(z) \longrightarrow \alpha_q^+(z), \qquad \inf_{z \in K} \alpha_q^+(z) > 0.$$

In contrast, (152) in Lemma 3.4 gives

$$r(c_r^+ + b_r^+ - z) = rc_r^+ + rb_r^+ - rz \longrightarrow 0$$

uniformly on K. Hence $C_r^+(z) > c_r^+ + b_r^+ - z$ for all sufficiently small r, and the indicator in (155) is zero. Thus the primary non-null contribution vanishes at $C_r^+(z)$. With $d = \{C_r^+(z) + z\}/(1 + r)$, (183) gives

$$C_r^+(z) \longrightarrow \infty, \qquad d - C_r^+(z) = \frac{z - rC_r^+(z)}{1 + r} \longrightarrow z - \alpha_q^+(z) > 0$$

uniformly on K. Hence Lemma 3.10 and (293) in Lemma A.2 give

$$0 \leq \bar{R}^+_{2,r,z}\{C_r^+(z)\} \leq \exp[-C_r^+(z)\{d - C_r^+(z)\}] \longrightarrow 0.$$

It follows that $\bar{R}^+_{0,r,z}\{C_r^+(z)\} \to 1$. By the definition of $D_r^+(z)$, uniformly on K,

$$D_r^+(z) = \bar{R}^+_{0,r,z}\{C_r^+(z)\} \longrightarrow 1.$$

□

3.5 Proofs of Theorem 3.1 and Propositions 3.2 and 3.3

Proof of Theorem 3.1. Choose a compact set

$$E \subset \mathbb{R} \setminus \{0, y_q^+\}$$

such that

$$\mathbb{E}\{D_+^*(Z)\mathbf{1}\{Z \in E\}\} > \ell_{\leq}(q) - \frac{\delta}{4}.$$

The proofs of Lemmas 3.11, 3.12, and 3.13 give, for all sufficiently small r, brackets

$$A_r(z) < B_r(z), \qquad z \in E,$$

and a constant $\gamma > 0$ such that

$$\sup_{z\in E} \sup_{u_q^+ \le c \le A_r(z)} \bar{R}^+_{r,z}(c) \le 1-\gamma, \qquad \inf_{z\in E} \bar{R}^+_{r,z}\{B_r(z)\} \ge 1+\gamma,$$

$$\sup_{z\in E} \sup_{A_r(z)\le c\le B_r(z)} \left|\bar{R}^+_{0,r,z}(c) - D^*_+(z)\right| \le \frac{\delta}{8}.$$

Fix such an r, and put $C_* = \sup_{z\in E} B_r(z) < \infty$. For $z \in E$, define

$$\mathcal{E}_N(z) = \max\left\{\|\widehat{R}_N - R^+_{r,z}\|_{\infty,[0,1]},\ \|\widehat{R}_{0,N} - \pi^+_{0,r}R^+_{0,r,z}\|_{\infty,[0,1]}\right\}$$

and

$$t_* = \frac{p_+(C_*)}{2q}\min\left\{\gamma, \frac{\delta}{8}\right\}.$$

Following the same argument in the proof of Theorem 2.8, Lemma 2.6 gives

$$\sup_{z\in E}\mathbb{P}\{\mathcal{E}_N(z) > t_* \mid Z = z\} \le 4e^{-2Nt_*^2}.$$

On the complementary event, the empirical and population curves differ by at most $q\mathcal{E}_N(z)/p_+(C_*) \le \frac{1}{2}\min\{\gamma, \delta/8\}$ throughout $[u_q^+, C_*]$. The two population brackets and Lemma 2.7 therefore imply

$$A_r(z) < \widehat{C}_N \le B_r(z), \qquad \mathrm{FDP} \ge D^*_+(z) - \frac{\delta}{4}.$$

Consequently,

$$\begin{aligned}
\mathbb{E}_{\Omega_N}\{\mathrm{FDR}_{\Omega_N}(\mathrm{BH}_q)\} &= \mathbb{E}_{\Omega_N,Z,\varepsilon}(\mathrm{FDP}) \\
&\ge \int_E \left\{D^*_+(z) - \frac{\delta}{4}\right\}\phi(z)\ \mathrm{d}z - 4e^{-2Nt_*^2} \\
&> \ell_{\le}(q) - \frac{\delta}{2} - 4e^{-2Nt_*^2}.
\end{aligned}$$

For all sufficiently large N, this gives

$$\mathbb{E}_{\Omega_N}\{\mathrm{FDR}_{\Omega_N}(\mathrm{BH}_q)\} > \ell_{\le}(q) - \frac{3\delta}{4}.$$

Therefore some deterministic realization ω satisfies the same inequality. The one-sided analogue of Lemma 2.9 permits an $\eta > 0$ for which the perturbed model has a positive definite correlation matrix and

$$\mathrm{FDR}^{(\eta)}_\omega(\mathrm{BH}_q) > \ell_{\le}(q) - \delta.$$

All means and null labels are unchanged by the perturbation. Table 4 gives the pre-perturbation sign pattern, and (33) shows that every off-diagonal covariance is multiplied by the positive factor $1-\eta^2$. Thus the perturbation preserves that sign pattern. □

Proof of Proposition 3.2. Since $e^{-(y_q^+)^2/2} = q$, the Mills expansion in Lemma A.2 gives

$$\overline{\Phi}(y_q^+) \sim \frac{q}{\sqrt{2\pi}\, y_q^+} = o(q).$$

Substitute $y_q^+ = \sqrt{2\log(1/q)}$ into (141). □

Proof of Proposition 3.3. By (143), $D^*_+(z) = q$ for $z \le 0$ and $D^*_+(z) > q$ for every $z > 0$. Hence $\ell_{\le}(q) = \mathbb{E}\{D^*_+(Z)\} > q$. □

4 Upper bounds

4.1 Preliminaries

The upper bounds concern the common-factor Gaussian model

$$T_i = \theta_i + a_i Z + \sqrt{1 - a_i^2}\,\varepsilon_i, \quad 1 \le i \le m, \tag{189}$$

where $Z, \varepsilon_1, \dots, \varepsilon_m$ are i.i.d. standard normal random variables and $|a_i| \le 1$. Its covariance matrix is diagonal plus rank one. Conditional on Z, the coordinates and their corresponding p-values are independent.

The following conditional leave-one-out bound will be used for both types of tests. Let $P_1, \dots, P_m$ be conditionally independent given Z, let $\mathcal{H}_0$ denote the true-null indices, and apply BH at level q. For $i \in \mathcal{H}_0$, write $F_{i,z}(t) = \mathbb{P}(P_i \le t \mid Z = z)$.

Lemma 4.1. *For almost every z,*

$$\mathbb{E}(\mathrm{FDP} \mid Z = z) \le \frac{q}{m} \sum_{i \in \mathcal{H}_0} \sup_{0 < t \le q} \frac{F_{i,z}(t)}{t} \tag{190}$$

Proof. Let R be the BH rejection count. For $i \in \mathcal{H}_0$, let R_i be the rejection count after replacing P_i by zero; in particular, $R_i \ge 1$. The standard leave-one-out argument [Benjamini and Yekutieli, 2001] starts from the pointwise identity

$$\frac{\mathbf{1}\{i \text{ is rejected}\}}{R \vee 1} = \frac{\mathbf{1}\{P_i \le qR_i/m\}}{R_i}.$$

Indeed, if i is rejected, replacing P_i by zero does not change the rejection count. Conversely, if $P_i \le qR_i/m$, restoring P_i leaves exactly R_i rejections by monotonicity and maximality of the BH step-up rule. Taking conditional expectations given $Z = z$ and summing gives, for almost every z,

$$\mathbb{E}(\mathrm{FDP} \mid Z = z) = \sum_{i \in \mathcal{H}_0} \mathbb{E}\left[\frac{\mathbf{1}\{P_i \le qR_i/m\}}{R_i} \,\middle|\, Z = z \right]. \tag{191}$$

Conditional independence implies that P_i and R_i are independent given $Z = z$. Since $0 < qR_i/m \le q$, conditioning gives, for every $i \in \mathcal{H}_0$ and almost every z,

$$\begin{aligned} \mathbb{E}\left[\frac{\mathbf{1}\{P_i \le qR_i/m\}}{R_i} \,\middle|\, Z = z \right] &= \frac{q}{m} \mathbb{E}\left[\frac{F_{i,z}(qR_i/m)}{qR_i/m} \,\middle|\, Z = z \right] \\ &\le \frac{q}{m} \sup_{0 < t \le q} \frac{F_{i,z}(t)}{t}. \end{aligned} \tag{192}$$

Summing (192) over $i \in \mathcal{H}_0$ proves (190). $\square$

4.2 Upper bound for two-sided tests

The common-factor class (189) contains the lower-bound construction in Subsection 2.4. The following theorem shows that this construction attains the sharp leading constant as $q \downarrow 0$. All constants below are universal and all asymptotic statements are uniform over the common-factor model.

Theorem 4.2. *For ordinary BH applied to the two-sided p-values $P_i = p(|T_i|)$, let $\mathcal{H}_0 = \{i : \theta_i = 0\}$, $m_0 = |\mathcal{H}_0|$, and $\pi_0 = m_0/m$. Throughout $0 < q \le 2\overline{\Phi}(1)$, let u_q be defined by $p(u_q) = 2\overline{\Phi}(u_q) = q$. If $m_0 = 0$, then $\mathrm{FDR}(\mathrm{BH}_q) = 0$. If $m_0 > 0$, then the following bounds hold uniformly over the common-factor model.*

(a)

$$\mathrm{FDR}(\mathrm{BH}_q) \le q\left\{1 + \pi_0\left(\frac{1-q}{2} + \frac{u_q}{\sqrt{2\pi}}\right)\right\}.$$

(b) There is a universal constant $C < \infty$ such that

$$\mathrm{FDR}(\mathrm{BH}_q) \le q\left[1 + \frac{\pi_0}{2\sqrt{\pi}}\sqrt{\log(1/q)} + C\pi_0\{\log(1/q)\}^{1/4}\right].$$

4.2.1 Proof of Theorem 4.2 (a)

Proof. If $m_0 = 0$, then $V = 0$ identically and hence $\mathrm{FDR} = 0$. Assume henceforth that $m_0 \ge 1$. Conditional on $Z = z$, each P_i is a function of ε_i alone, or is deterministic when $|a_i| = 1$. Thus the p-values are conditionally independent, and Lemma 4.1 gives (190).

To average (190) over Z, it remains to control

$$\int_{-u_q}^{u_q} \phi(z) \sup_{0<t\le q} \frac{F_{i,z}(t)}{t}\,\mathrm{d}z$$

for each true null i. We do this on $|z| \le u_q$; the event $|Z| > u_q$ is handled at the end by $\mathrm{FDP} \le 1$. Fix a true null index i. Put $r = |a_i|$ and $s = \sqrt{1-r^2}$, and first suppose $s > 0$. Conditional on $Z = z$, $T_i \sim N(a_i z, s^2)$. The inequality $t = p(u) = 2\overline{\Phi}(u) \le q$ holds if and only if $u \ge u_q$, and in this case

$$F_{i,z}(p(u)) = \int_u^\infty s^{-1}\phi\{(x-a_i z)/s\}\,\mathrm{d}x + \int_u^\infty s^{-1}\phi\{(x+a_i z)/s\}\,\mathrm{d}x.$$

Since $s^2 = 1 - a_i^2$,

$$\begin{aligned}
\frac{s^{-1}\phi\{(x-a_i z)/s\}}{\phi(x)} &= \frac{1}{s}\exp\left\{-\frac{(x-a_i z)^2}{2s^2} + \frac{x^2}{2}\right\} \\
&= \frac{1}{s}\exp\left\{\frac{z^2}{2} - \frac{(a_i x - z)^2}{2s^2}\right\}.
\end{aligned}$$

The second tail is the same calculation with z replaced by $-z$. Choose $\eta_+, \eta_- \in \{-1, 1\}$ with $\{\eta_+, \eta_-\} = \{-1, 1\}$ so that $\eta_+ a_i z \ge 0$ and $\eta_- a_i z \le 0$. For $x \ge u$,

$$|a_i x - \eta_+ z| \ge (ru - |z|)_+, \qquad |a_i x - \eta_- z| \ge ru + |z|.$$

Both right-hand sides are nondecreasing in u, so for $u \ge u_q$,

$$\begin{aligned}
\phi(z)\frac{F_{i,z}(p(u))}{p(u)} &\le \frac{\phi(z)}{p(u)}\frac{e^{z^2/2}}{s}\left[e^{-(ru_q-|z|)_+^2/(2s^2)} + e^{-(ru_q+|z|)^2/(2s^2)}\right]\int_u^\infty \phi(x)\,\mathrm{d}x \\
&= \frac{1}{2s\sqrt{2\pi}}\left[e^{-(ru_q-|z|)_+^2/(2s^2)} + e^{-(ru_q+|z|)^2/(2s^2)}\right],
\end{aligned}$$

because $\int_u^\infty \phi(x)\,\mathrm{d}x = p(u)/2$. Taking the supremum gives

$$\phi(z)\sup_{0<t\le q}\frac{F_{i,z}(t)}{t} \le \frac{1}{2s\sqrt{2\pi}}\left[e^{-(ru_q-|z|)_+^2/(2s^2)} + e^{-(ru_q+|z|)^2/(2s^2)}\right]. \tag{193}$$

Integrating (193) over $|z| \le u_q$ and using symmetry gives

$$\int_{-u_q}^{u_q} \phi(z) \sup_{0<t\le q} \frac{F_{i,z}(t)}{t} \,\mathrm{d}z \le \frac{1}{s\sqrt{2\pi}} \left\{ \int_0^{u_q} e^{-(ru_q-z)_+^2/(2s^2)} \,\mathrm{d}z + \int_0^{u_q} e^{-(ru_q+z)^2/(2s^2)} \,\mathrm{d}z \right\}. \tag{194}$$

The two integrals on the right-hand side of (194) are

$$\begin{aligned} \int_0^{u_q} e^{-(ru_q-z)_+^2/(2s^2)} \,\mathrm{d}z &= \int_0^{ru_q} e^{-y^2/(2s^2)} \,\mathrm{d}y + (1-r)u_q, \\ \int_0^{u_q} e^{-(ru_q+z)^2/(2s^2)} \,\mathrm{d}z &= \int_{ru_q}^{(1+r)u_q} e^{-y^2/(2s^2)} \,\mathrm{d}y. \end{aligned}$$

Consequently, writing $\gamma_r = \sqrt{(1-r)/(1+r)} \in (0,1]$,

$$\begin{aligned} \int_{-u_q}^{u_q} \phi(z) \sup_{0<t\le q} \frac{F_{i,z}(t)}{t} \,\mathrm{d}z &\le \frac{1}{s\sqrt{2\pi}} \left\{ \int_0^{(1+r)u_q} e^{-y^2/(2s^2)} \,\mathrm{d}y + (1-r)u_q \right\} \\ &= \left\{ \Phi\left(\frac{(1+r)u_q}{s}\right) - \frac{1}{2} \right\} + \frac{(1-r)u_q}{s\sqrt{2\pi}} \\ &= \Phi(u_q/\gamma_r) - \frac{1}{2} + \frac{u_q\gamma_r}{\sqrt{2\pi}}, \end{aligned} \tag{195}$$

where the final equality in (195) uses $s^2 = 1 - r^2 = (1-r)(1+r)$, so $(1+r)/s = 1/\gamma_r$ and $(1-r)/s = \gamma_r$. The right-hand side of (195) is increasing in $\gamma_r \in (0,1]$. Its derivative with respect to γ_r is

$$\frac{u_q}{\sqrt{2\pi}} \left[1 - \gamma_r^{-2} \exp\{-u_q^2/(2\gamma_r^2)\} \right] \ge 0,$$

because $u_q \ge 1$ and, on putting $w = \gamma_r^{-2} \ge 1$,

$$we^{-u_q^2 w/2} \le we^{-w/2} \le 2/e < 1.$$

Hence (195) is at most

$$\Phi(u_q) - \frac{1}{2} + \frac{u_q}{\sqrt{2\pi}} = \frac{1-q}{2} + \frac{u_q}{\sqrt{2\pi}}. \tag{196}$$

When $r = 1$, the conditional p-value is deterministic. For $|z| < u_q$, it exceeds q, and the boundary $|z| = u_q$ has Lebesgue measure zero; hence (196) remains valid. When $r = 0$, the conditional null p-value is uniform and independent of Z; (196) remains valid, although it is not sharp.

Let $A = \{|Z| \le u_q\}$. Decomposing over A and A^c,

$$\mathrm{FDR} = \mathbb{E}(\mathrm{FDP}\,\mathbf{1}_A) + \mathbb{E}(\mathrm{FDP}\,\mathbf{1}_{A^c}).$$

For the first term, integrate (190) over $z \in [-u_q, u_q]$ and use (196). For the second term, no conditional tail bound is needed: since $0 \le \mathrm{FDP} \le 1$,

$$\mathbb{E}(\mathrm{FDP}\,\mathbf{1}_{A^c}) \le \mathbb{P}(A^c) = \mathbb{P}(|Z| > u_q) = 2\overline{\Phi}(u_q) = q. \tag{197}$$

Combining (190), (196), and (197) yields

$$\mathrm{FDR} \le q + \frac{q}{m} \sum_{i\in\mathcal{H}_0} \left(\frac{1-q}{2} + \frac{u_q}{\sqrt{2\pi}} \right) = q \left\{ 1 + \pi_0 \left(\frac{1-q}{2} + \frac{u_q}{\sqrt{2\pi}} \right) \right\},$$

as claimed. □

4.2.2 Proof of Theorem 4.2 (b)

The proof of part (b) has two parts. First we prove a deterministic statement for the rank-weighted contribution that will arise from the finite BH threshold. We then verify its hypotheses from the finite threshold by a uniform concentration argument for Poisson–binomial counts, that is, sums of independent Bernoulli random variables with possibly different success probabilities. Throughout the proof, C and c denote positive universal constants whose values may change from line to line.

For $0 \le r < 1$, define

$$\rho_r(u, z) = \frac{\overline{\Phi}\{(u - rz)/\sqrt{1 - r^2}\} + \overline{\Phi}\{(u + rz)/\sqrt{1 - r^2}\}}{2\overline{\Phi}(u)}. \tag{198}$$

For $r = 1$, set $\rho_1(u, z) = \mathbf{1}\{|z| \ge u\}/p(u)$; the convention on the measure-zero boundary $|z| = u$ is immaterial. For every true null i, this is its conditional two-sided tail ratio:

$$\frac{F_{i,z}\{p(u)\}}{p(u)} = \rho_{|a_i|}(u, z), \tag{199}$$

because the two Gaussian tails are unchanged when a_i is replaced by $|a_i|$. Integrating (199) over the common factor gives the unconditional normalization: for every true null i and every $u > 0$,

$$\int_{\mathbb{R}} \rho_{|a_i|}(u, z)\phi(z)\,\mathrm{d}z = \frac{1}{p(u)} \int_{\mathbb{R}} F_{i,z}\{p(u)\}\phi(z)\,\mathrm{d}z = \frac{\mathbb{P}(P_i \le p(u))}{p(u)} = 1. \tag{200}$$

The middle equality follows because

$$F_{i,z}\{p(u)\} = \mathbb{P}(P_i \le p(u) \mid Z = z),$$

and Z has density ϕ. Hence

$$\int_{\mathbb{R}} F_{i,z}\{p(u)\}\phi(z)\,\mathrm{d}z = \mathbb{E}\{\mathbb{P}(P_i \le p(u) \mid Z)\} = \mathbb{P}(P_i \le p(u)),$$

which is the tower property. The last equality in (200) uses that P_i is a two-sided null p-value.

We first give a useful bound on $\rho_r(u, z)$ without restrictions on r.

Lemma 4.3. *If $u_q \ge 1$ and $0 \le r < 1$, then*

$$\int_{-u_q}^{u_q} \phi(z) \sup_{u \ge u_q} \rho_r(u, z)\,\mathrm{d}z \le \Phi\left(\frac{(1 + r)u_q}{\sqrt{1 - r^2}}\right) - \frac{1}{2} + \frac{(1 - r)u_q}{\sqrt{1 - r^2}\sqrt{2\pi}}.$$

For $r = 1$, the integral on the left is zero.

Proof. Assume first that $0 \le r < 1$. For $\xi \in \{-1, 1\}$, using $\phi(t) = (2\pi)^{-1/2}\exp(-t^2/2)$,

$$\begin{aligned}
\frac{\phi\{(x - \xi rz)/\sqrt{1 - r^2}\}}{\sqrt{1 - r^2}\,\phi(x)} &= \frac{1}{\sqrt{1 - r^2}} \exp\left\{-\frac{(x - \xi rz)^2}{2(1 - r^2)} + \frac{x^2}{2}\right\} \\
&= \frac{1}{\sqrt{1 - r^2}} \exp\left\{\frac{z^2}{2} - \frac{(rx - \xi z)^2}{2(1 - r^2)}\right\}.
\end{aligned}$$

The second equality uses the identity

$$(1 - r^2)z^2 - (rx - \xi z)^2 = (1 - r^2)x^2 - (x - \xi rz)^2 = -r^2x^2 + 2\xi rxz - r^2z^2.$$

If $\xi z = |z|$, then

$$|rx - \xi z| = |rx - |z|| \geq (rx - |z|)_+ \geq (ru_q - |z|)_+,$$

while if $\xi z = -|z|$, then

$$|rx - \xi z| = rx + |z| \geq ru_q + |z|.$$

For fixed $u \geq u_q$, each conditional tail in the numerator of $\rho_r(u, z)$ can be written as

$$\overline{\Phi}\{(u - \xi rz)/\sqrt{1-r^2}\} = \int_u^\infty \frac{\phi\{(x - \xi rz)/\sqrt{1-r^2}\}}{\sqrt{1-r^2}} \,\mathrm{d}x.$$

Multiplying the density-ratio identity above by $\phi(z)$ and using the two lower bounds on $|rx - \xi z|$, we get, for $|z| \leq u_q$,

$$\begin{aligned}
\phi(z)\rho_r(u, z) &= \frac{1}{p(u)} \sum_{\xi \in \{-1,1\}} \int_u^\infty \phi(x)\phi(z) \frac{\phi\{(x - \xi rz)/\sqrt{1-r^2}\}}{\sqrt{1-r^2}\,\phi(x)} \,\mathrm{d}x \\
&\leq \frac{\int_u^\infty \phi(x)\,\mathrm{d}x}{p(u)} \frac{1}{\sqrt{1-r^2}\sqrt{2\pi}} \left[e^{-(ru_q - |z|)_+^2/\{2(1-r^2)\}} + e^{-(ru_q + |z|)^2/\{2(1-r^2)\}} \right].
\end{aligned}$$

Since $p(u) = 2\int_u^\infty \phi(x)\,\mathrm{d}x$, and the right side no longer depends on u, taking the supremum over $u \geq u_q$ gives

$$\phi(z) \sup_{u \geq u_q} \rho_r(u, z) \leq \frac{1}{2\sqrt{1-r^2}\sqrt{2\pi}} \left[e^{-(ru_q - |z|)_+^2/\{2(1-r^2)\}} + e^{-(ru_q + |z|)^2/\{2(1-r^2)\}} \right].$$

Integrating this bound over $z \in [-u_q, u_q]$ and using symmetry reduces the right side to the integral over $0 \leq z \leq u_q$ without the factor $1/2$. The first exponential contributes

$$\begin{aligned}
\int_0^{u_q} e^{-(ru_q - z)_+^2/\{2(1-r^2)\}} \,\mathrm{d}z &= \int_0^{ru_q} e^{-(ru_q - z)^2/\{2(1-r^2)\}} \,\mathrm{d}z + \int_{ru_q}^{u_q} 1 \,\mathrm{d}z \\
&= \int_0^{ru_q} e^{-y^2/\{2(1-r^2)\}} \,\mathrm{d}y + (1-r)u_q,
\end{aligned}$$

where the last line uses $y = ru_q - z$ in the first integral. The second exponential contributes

$$\int_0^{u_q} e^{-(ru_q + z)^2/\{2(1-r^2)\}} \,\mathrm{d}z = \int_{ru_q}^{(1+r)u_q} e^{-y^2/\{2(1-r^2)\}} \,\mathrm{d}y,$$

using $y = ru_q + z$. Therefore

$$\begin{aligned}
&\int_{-u_q}^{u_q} \phi(z) \sup_{u \geq u_q} \rho_r(u, z)\,\mathrm{d}z \\
&\leq \frac{1}{\sqrt{1-r^2}\sqrt{2\pi}} \left\{ \int_0^{ru_q} e^{-y^2/\{2(1-r^2)\}} \,\mathrm{d}y + \int_{ru_q}^{(1+r)u_q} e^{-y^2/\{2(1-r^2)\}} \,\mathrm{d}y + (1-r)u_q \right\} \\
&= \Phi\{(1+r)u_q/\sqrt{1-r^2}\} - \frac{1}{2} + \frac{(1-r)u_q}{\sqrt{1-r^2}\sqrt{2\pi}},
\end{aligned}$$

which is the asserted bound.

When $r = 1$, the conditional residual variance $1 - r^2$ is zero. Thus a null statistic with $|a_i| = 1$ is $T_i = a_i Z$, so conditional on $Z = z$ it equals $\pm z$ and has absolute value $|z|$. For $|z| < u_q$ it cannot cross any threshold $u \geq u_q$. On the integration range $[-u_q, u_q]$, equality $|z| = u$ for some $u \geq u_q$ can occur only when $|z| = u = u_q$, that is, at the two points $z = \pm u_q$. This set has Lebesgue measure zero, so the convention used for $\rho_1(u, z)$ on the boundary does not change the integral.

□

Define

$$K_q = \sqrt{8 \log u_q}, \qquad L_q = K_q + \sqrt{u_q}. \tag{201}$$

The following lemma shows a refined bound on $\rho_r(u, z)$ for small r.

Lemma 4.4. *With K_q and L_q as defined in (201), uniformly for $0 \le r \le 3/5$, $u \ge u_q$, and $|z| \le u_q$,*

$$\phi(z)\rho_r(u, z) \le C \left[e^{-(z-ru)^2/\{2(1-r^2)\}} + e^{-(z+ru)^2/\{2(1-r^2)\}} \right]. \tag{202}$$

Moreover, there are universal constants $C < \infty$ and $q_0 > 0$ such that, whenever $q \le q_0$, if $|z| \ge L_q$ and $||z| - ru| \le K_q$, then

$$\phi(z)\rho_r(u, z) \le \frac{1 + C(K_q + 1)/u_q}{2\sqrt{2\pi}\sqrt{1-r^2}} + C \exp\{-u_q\}. \tag{203}$$

Proof. Since $r \le 3/5$, $u \ge u_q$, and $|z| \le u_q$,

$$u - rz \ge u - r|z| \ge u - ru_q \ge (1-r)u \ge \frac{2}{5}u \ge \frac{2}{5}u_q.$$

Since $\sqrt{1-r^2} \le 1$, this gives

$$\frac{u - rz}{\sqrt{1-r^2}} \ge \frac{2}{5}u_q.$$

Hence the upper Mills bound (305) in Appendix A applies to $(u - rz)/\sqrt{1-r^2}$, while the lower Mills bound (306) applies to u. These two inequalities give

$$\begin{aligned}
\frac{\phi(z)\overline{\Phi}\{(u-rz)/\sqrt{1-r^2}\}}{2\overline{\Phi}(u)} &\le \frac{\phi(z)\phi\{(u-rz)/\sqrt{1-r^2}\}}{2\phi(u)} \frac{\sqrt{1-r^2}(u^2+1)}{u(u-rz)} \\
&= \frac{1}{2\sqrt{2\pi}} \frac{\sqrt{1-r^2}(u^2+1)}{u(u-rz)} \exp\left\{ -\frac{z^2}{2} - \frac{(u-rz)^2}{2(1-r^2)} + \frac{u^2}{2} \right\} \\
&= \frac{1}{2\sqrt{2\pi}} \frac{\sqrt{1-r^2}(u^2+1)}{u(u-rz)} \exp\left\{ -\frac{(1-r^2)z^2 + (u-rz)^2 - (1-r^2)u^2}{2(1-r^2)} \right\} \\
&= \frac{1}{2\sqrt{2\pi}} \frac{\sqrt{1-r^2}(u^2+1)}{u(u-rz)} \exp\left\{ -\frac{(z-ru)^2}{2(1-r^2)} \right\}.
\end{aligned} \tag{204}$$

Using a similar argument, we can prove

$$\frac{\phi(z)\overline{\Phi}\{(u+rz)/\sqrt{1-r^2}\}}{2\overline{\Phi}(u)} \le \frac{1}{2\sqrt{2\pi}} \frac{\sqrt{1-r^2}(u^2+1)}{u(u+rz)} \exp\left\{ -\frac{(z+ru)^2}{2(1-r^2)} \right\}. \tag{205}$$

For both signs, $|z| \le u_q \le u$ implies

$$u \mp rz \ge u - r|z| \ge (1-r)u \ge \frac{2}{5}u.$$

Also $\sqrt{1-r^2} \le 1$ and $u \ge u_q \ge 1$. Therefore

$$\max\left\{ \frac{\sqrt{1-r^2}(u^2+1)}{u(u-rz)}, \frac{\sqrt{1-r^2}(u^2+1)}{u(u+rz)} \right\} \le \frac{u^2+1}{(2/5)u^2} \le 5.$$

This bound is uniform over $0 \le r \le 3/5$, $u \ge u_q$, and $|z| \le u_q$. Substituting it into (204) and (205), and increasing C, proves (202). If $|z| = ru + d$ with $|d| \le K_q$, then the first term in (204)–(205) has prefactor

$$\frac{\sqrt{1-r^2}(u^2+1)}{u\{(1-r^2)u - rd\}} = \frac{1}{\sqrt{1-r^2}} \frac{1+u^{-2}}{1 - rd/\{(1-r^2)u\}}. \tag{206}$$

Because $r \le 3/5$, one has $1 - r^2 \ge 16/25$. Hence, when $|d| \le K_q \le u_q/4$,

$$\left|\frac{rd}{(1-r^2)u}\right| \le \frac{(3/5)K_q}{(16/25)u_q} \le \frac{15}{64}.$$

Since $(1-x)^{-1} \le 1 + C|x|$ for $|x| \le 15/64$, and since $u \ge u_q \ge 1$,

$$\frac{1+u^{-2}}{1 - rd/\{(1-r^2)u\}} \le (1 + u_q^{-1}) \left(1 + C\frac{K_q}{u_q}\right) \le 1 + C\frac{K_q+1}{u_q}.$$

Combining this estimate with (206) gives

$$\begin{aligned}
\frac{\sqrt{1-r^2}(u^2+1)}{u\{(1-r^2)u - rd\}} &= \frac{1}{\sqrt{1-r^2}} \frac{1+u^{-2}}{1 - rd/\{(1-r^2)u\}} \\
&\le \frac{1}{\sqrt{1-r^2}} \left\{1 + C\frac{K_q+1}{u_q}\right\} \\
&= \frac{1 + C(K_q+1)/u_q}{\sqrt{1-r^2}}.
\end{aligned}$$

Note that

$$\begin{aligned}
z \ge 0: &\quad |z + ru| = |z| + ru = 2|z| - d \ge 2L_q - K_q, \\
z < 0: &\quad |z - ru| = |z| + ru = 2|z| - d \ge 2L_q - K_q.
\end{aligned}$$

Therefore, with the $+$ sign when $z \ge 0$ and the $-$ sign when $z < 0$,

$$\exp\left\{-\frac{(z \pm ru)^2}{2(1-r^2)}\right\} \le \exp\left\{-\frac{(2L_q - K_q)^2}{2(1-r^2)}\right\} \le \exp\{-u_q\}, \tag{207}$$

because $1 - r^2 \le 1$ and $(2L_q - K_q)^2/2 \ge u_q$. The proof is then completed by (204) - (207). $\square$

For the next three lemmas, fix $0 < r \le 3/5$ and use K_q, L_q from (201). Let Π be a positive measure of mass at most one on $(\theta, \alpha) \in \mathbb{R} \times [-1, 1]$, let $\varepsilon \sim N(0,1)$, and define

$$G_z(u) = \int \mathbb{P}\{|\theta + \alpha z + \sqrt{1-\alpha^2}\,\varepsilon| \ge u\}\ \mathrm{d}\Pi(\theta, \alpha). \tag{208}$$

For z and u, write

$$\begin{aligned}
G_z^{\text{large}}(u) &= \int_{\{|\alpha| \ge r/u_q\}} \mathbb{P}\{|\theta + \alpha z + \sqrt{1-\alpha^2}\,\varepsilon| \ge u\}\ \mathrm{d}\Pi(\theta, \alpha), \\
G_z^{\text{small}}(u) &= \int_{\{|\alpha| < r/u_q\}} \mathbb{P}\{|\theta + \alpha z + \sqrt{1-\alpha^2}\,\varepsilon| \ge u\}\ \mathrm{d}\Pi(\theta, \alpha).
\end{aligned}$$

Then

$$G_z(u) = G_z^{\text{large}}(u) + G_z^{\text{small}}(u).$$

For each z, let $\mathcal{C}_z$ be a finite set satisfying

$$\mathcal{C}_z \subseteq \left\{ u \geq u_q : \; G_z(u) \geq \frac{p(u)}{2q}, \; G_z(v) \leq 3\frac{p(v)}{q} \; \forall \, u_q \leq v < u \right\}. \tag{209}$$

Define

$$\begin{aligned} E_{\pm}^{\text{large}}(r) = \Big\{ z : \; \pm z \in [L_q, u_q], \; \exists u \text{ such that } u \in \mathcal{C}_z, \; \big||z| - ru\big| \leq K_q, \\ \text{and } G_z^{\text{large}}(u) \geq p(u)/(4q) \Big\}, \end{aligned} \tag{210}$$

and

$$\begin{aligned} E_{\pm}^{\text{small}}(r) = \Big\{ z : \; \pm z \in [L_q, u_q], \; \exists u \text{ such that } u \in \mathcal{C}_z, \; \big||z| - ru\big| \leq K_q, \\ \text{and } G_z^{\text{small}}(u) \geq p(u)/(4q) \Big\}. \end{aligned} \tag{211}$$

The next lemma bounds $|E_{\pm}^{\text{large}}(r)|$.

Lemma 4.5. *There are universal constants $C < \infty$ and $q_0 > 0$ such that, whenever $q \leq q_0$, the following bound holds uniformly over all positive measures Π on $\mathbb{R} \times [-1, 1]$ with $\Pi(\mathbb{R} \times [-1, 1]) \leq 1$, and over all families $(\mathcal{C}_z)$ satisfying* (209)*:*

$$|E_{+}^{\text{large}}(r)| + |E_{-}^{\text{large}}(r)| \leq (1 - r)u_q + C\sqrt{u_q}. \tag{212}$$

Proof. Set

$$\begin{aligned} Y_z = Y_z(\theta, \alpha, e) &= \theta + \alpha z + \sqrt{1 - \alpha^2}\, e, \\ \mathrm{d}\nu(\theta, \alpha, e) &= \frac{1}{3}\, \mathrm{d}\Pi(\theta, \alpha)\phi(e)\, \mathrm{d}e. \end{aligned} \tag{213}$$

Then

$$\begin{aligned} \nu\{|Y_z| \geq v\} &= \frac{1}{3} G_z(v), \\ \nu\{|\alpha| \geq r/u_q, \; |Y_z| \geq v\} &= \frac{1}{3} G_z^{\text{large}}(v). \end{aligned} \tag{214}$$

If $u \in \mathcal{C}_z$ and $u_q \leq v < u$, then (209) gives

$$\nu\{|Y_z| \geq v\} \leq \frac{p(v)}{q}. \tag{215}$$

If $z \in E_{+}^{\text{large}}(r)$ and $u \in \mathcal{C}_z$ is as in (210), then by (214),

$$\nu\{|\alpha| \geq r/u_q, \; |Y_z| \geq u\} \geq \frac{p(u)}{12q}. \tag{216}$$

The same result holds with $z \in E_{-}^{\text{large}}(r)$ and $u \in \mathcal{C}_z$.

Fix any $z_1 \in E_{+}^{\text{large}}(r)$. Consider $z_2 \in E_{+}^{\text{large}}(r)$ and $-z_3 \in E_{-}^{\text{large}}(r)$ with $z_2 > z_1$. Choose $v_1 \in \mathcal{C}_{z_1}$, $v_2 \in \mathcal{C}_{z_2}$, and $v_3 \in \mathcal{C}_{-z_3}$ as in (210), and put

$$d_2 = \frac{r}{u_q}(z_2 - z_1), \qquad d_3 = \frac{r}{u_q}(z_1 + z_3). \tag{217}$$

Split $\{|\alpha| \geq r/u_q, \; |Y_{z_1}| \geq v_1\} = A_+ \dot\cup A_-$, where

$$\begin{aligned} A_+ &= \{|\alpha| \geq r/u_q, \; |Y_{z_1}| \geq v_1, \; \alpha Y_{z_1} \geq 0\}, \\ A_- &= \{|\alpha| \geq r/u_q, \; |Y_{z_1}| \geq v_1, \; \alpha Y_{z_1} < 0\}. \end{aligned}$$

Then (216) gives

$$\nu(A_+) + \nu(A_-) = \nu\{|\alpha| \geq r/u_q,\ |Y_{z_1}| \geq v_1\} \geq \frac{p(v_1)}{12q}.$$

Moreover,

$$A_+ \subseteq \{|Y_{z_2}| \geq v_1 + d_2\}, \qquad A_- \subseteq \{|Y_{-z_3}| \geq v_1 + d_3\}.$$

Indeed, on A_+,

$$|Y_{z_2}| = |Y_{z_1} + \alpha(z_2 - z_1)| \geq |Y_{z_1}| + |\alpha|(z_2 - z_1) \geq v_1 + d_2,$$

while on A_-,

$$|Y_{-z_3}| = |Y_{z_1} - \alpha(z_1 + z_3)| \geq |Y_{z_1}| + |\alpha|(z_1 + z_3) \geq v_1 + d_3.$$

Let

$$\widetilde{E}_-^{\text{large}}(r) = \{-z :\ z \in E_-^{\text{large}}(r)\}.$$

It is enough to bound $|E_+^{\text{large}}(r)| + |\widetilde{E}_-^{\text{large}}(r)|$. If $E_+^{\text{large}}(r) \cup \widetilde{E}_-^{\text{large}}(r)$ is empty, there is nothing to prove. Otherwise choose $z_1 \in E_+^{\text{large}}(r) \cup \widetilde{E}_-^{\text{large}}(r)$ arbitrarily close to

$$\inf\{z :\ z \in E_+^{\text{large}}(r) \cup \widetilde{E}_-^{\text{large}}(r)\}.$$

Choose $v_1 \in \mathcal{C}_{z_1}$ if $z_1 \in E_+^{\text{large}}(r)$, and $v_1 \in \mathcal{C}_{-z_1}$ if $z_1 \in \widetilde{E}_-^{\text{large}}(r)$. In the first case, (210) gives

$$z_1 \in [L_q, u_q], \qquad |z_1 - rv_1| \leq K_q.$$

In the second case, $-z_1 \in E_-^{\text{large}}(r)$, so the same definition gives

$$z_1 = |-z_1| \in [L_q, u_q], \qquad |z_1 - rv_1| \leq K_q.$$

Thus, in either case,

$$z_1 \geq L_q, \quad |z_1 - rv_1| \leq K_q \quad \Longrightarrow \quad \frac{rv_1}{u_q} \geq \frac{z_1 - K_q}{u_q} \geq \frac{L_q - K_q}{u_q}. \tag{218}$$

Set

$$B_q = \frac{2K_q + 2r^2}{1 - r^2/u_q} + \frac{4u_q}{L_q - K_q}. \tag{219}$$

We claim that, to the right of $z_1 + B_q$, the two reflected sets cannot overlap:

$$\big(E_+^{\text{large}}(r) \cap (z_1 + B_q, u_q]\big) \cap \big(\widetilde{E}_-^{\text{large}}(r) \cap (z_1 + B_q, u_q]\big) = \emptyset. \tag{220}$$

We prove (220) by contradiction. Assume (220) fails. Then there exist

$$z_2 \in E_+^{\text{large}}(r), \qquad z_3 \in \widetilde{E}_-^{\text{large}}(r), \qquad z_2, z_3 > z_1 + B_q. \tag{221}$$

If $z_1 \in E_+^{\text{large}}(r)$, assume (221) and choose $v_2 \in \mathcal{C}_{z_2}$ and $v_3 \in \mathcal{C}_{-z_3}$. By (210),

$$|z_1 - rv_1| \leq K_q, \qquad |z_2 - rv_2| \leq K_q, \qquad |z_3 - rv_3| \leq K_q,$$

where the last inequality uses $-z_3 \in E_-^{\text{large}}(r)$, so that $|-z_3| = z_3$. Hence

$$\begin{aligned} r(v_2 - v_1) &= (rv_2 - z_2) + (z_2 - z_1) + (z_1 - rv_1) \geq z_2 - z_1 - 2K_q, \\ r(v_3 - v_1) &= (rv_3 - z_3) + (z_3 - z_1) + (z_1 - rv_1) \geq z_3 - z_1 - 2K_q. \end{aligned}$$

Therefore

$$v_2 - v_1 \geq \frac{z_2 - z_1 - 2K_q}{r}, \qquad v_3 - v_1 \geq \frac{z_3 - z_1 - 2K_q}{r}. \tag{222}$$

Since $z_2, z_3 > z_1 + B_q$,

$$z_2 - z_1,\ z_3 - z_1 > \frac{2K_q + 2r^2}{1 - r^2/u_q}.$$

Therefore

$$(z_2 - z_1)\left(1 - \frac{r^2}{u_q}\right) > 2K_q,$$

and, using $z_1 \leq u_q$,

$$(z_3 - z_1)\left(1 - \frac{r^2}{u_q}\right) > 2K_q + 2r^2 \geq 2K_q + \frac{2r^2 z_1}{u_q}.$$

Equivalently,

$$\frac{z_2 - z_1 - 2K_q}{r} > \frac{r}{u_q}(z_2 - z_1), \qquad \frac{z_3 - z_1 - 2K_q}{r} > \frac{r}{u_q}(z_1 + z_3).$$

Combining these inequalities with (222) gives

$$v_2 - v_1 > \frac{r}{u_q}(z_2 - z_1), \qquad v_3 - v_1 > \frac{r}{u_q}(z_1 + z_3),$$

that is,

$$v_1 + d_2 < v_2, \qquad v_1 + d_3 < v_3. \tag{223}$$

Also, (218) gives

$$\frac{rv_1}{u_q} \geq \frac{L_q - K_q}{u_q}.$$

Since $z_2, z_3 > z_1 + B_q$ and the second term in B_q is $4u_q/(L_q - K_q)$,

$$z_2 - z_1 > \frac{4u_q}{L_q - K_q}, \qquad z_1 + z_3 > z_3 - z_1 > \frac{4u_q}{L_q - K_q}.$$

Consequently

$$\frac{rv_1}{u_q}\min\{z_2 - z_1, z_1 + z_3\} \geq \frac{L_q - K_q}{u_q}\min\{z_2 - z_1, z_1 + z_3\} > 4$$

and hence, by (217),

$$\min\{d_2, d_3\} > \frac{4}{v_1}. \tag{224}$$

The ordering in (223), together with $v_1 \geq u_q$ and $d_2, d_3 \geq 0$, gives

$$u_q \leq v_1 + d_2 < v_2, \qquad u_q \leq v_1 + d_3 < v_3. \tag{225}$$

Thus $v_2 \in \mathcal{C}_{z_2}$, $v_3 \in \mathcal{C}_{-z_3}$, so (215) applies with $(z, u, v) = (z_2, v_2, v_1 + d_2)$ and $(z, u, v) = (-z_3, v_3, v_1 + d_3)$. Hence

$$\begin{aligned} \frac{p(v_1)}{12q} &\leq \nu(A_+) + \nu(A_-) \\ &\leq \nu\{|Y_{z_2}| \geq v_1 + d_2\} + \nu\{|Y_{-z_3}| \geq v_1 + d_3\} \\ &\leq \frac{p(v_1 + d_2) + p(v_1 + d_3)}{q}, \qquad \text{by (215).} \end{aligned}$$

Multiplying by q and substituting the definitions of d_2 and d_3 in (217) imply

$$\frac{p(v_1)}{12} \leq p\{v_1 + (r/u_q)(z_2 - z_1)\} + p\{v_1 + (r/u_q)(z_1 + z_3)\}. \tag{226}$$

For $u > 0$ and $d \geq 0$, $p = 2\overline{\Phi}$ and the normal hazard $\lambda(t) = \phi(t)/\overline{\Phi}(t)$ give

$$\begin{aligned} \frac{p(u+d)}{p(u)} &= \frac{\overline{\Phi}(u+d)}{\overline{\Phi}(u)} = \exp\left\{-\int_u^{u+d} \lambda(t)\ \mathrm{d}t\right\} \\ &\leq \exp\left\{-\int_u^{u+d} t\ \mathrm{d}t\right\} = \exp\{-ud - d^2/2\} \leq e^{-ud}, \end{aligned} \tag{227}$$

where the inequality uses $\lambda(t) > t$ for $t > 0$, as stated in (293) of Appendix Lemma A.2. Thus the right side of (226) is at most

$$\frac{p(v_1)}{48} + \frac{p(v_1)}{48} < \frac{p(v_1)}{12},$$

which is impossible. Equivalently, the tail calculation gives

$$\frac{r}{u_q} \min\{z_2 - z_1, z_1 + z_3\} = \min\{d_2, d_3\} \leq \frac{4}{v_1}. \tag{228}$$

This contradicts (224). By the same argument with signs reversed, if $-z_1, -z_3 \in E_-^{\text{large}}(r)$, $z_2 \in E_+^{\text{large}}(r)$, and $z_3 > z_1$, then, for $v_1 \in \mathcal{C}_{-z_1}$ chosen as in (210),

$$\frac{r}{u_q} \min\{z_3 - z_1, z_1 + z_2\} \leq \frac{4}{v_1}. \tag{229}$$

If $z_1 \in \widetilde{E}_-^{\text{large}}(r)$, the same contradiction uses (229). Together with the preceding case $z_1 \in E_+^{\text{large}}(r)$, this proves (220).

Put

$$a = \inf\{z : \ z \in E_+^{\text{large}}(r) \cup \widetilde{E}_-^{\text{large}}(r)\}.$$

Since z_1 can be chosen arbitrarily close to a, (220) gives, after letting $z_1 \downarrow a$,

$$\big(E_+^{\text{large}}(r) \cap (a + B_q, u_q]\big) \cap \big(\widetilde{E}_-^{\text{large}}(r) \cap (a + B_q, u_q]\big) = \emptyset.$$

By (210) and (209), both sets are contained in

$$I_q = [\max\{L_q, ru_q - K_q\}, u_q].$$

Indeed, for any signed point $s \in E_+^{\text{large}}(r) \cup E_-^{\text{large}}(r)$, and for its associated $u \in \mathcal{C}_s$,

$$|s| \in [L_q, u_q], \qquad |s| \geq ru - K_q \geq ru_q - K_q.$$

Recall the definition of B_q in (219). Set

$$J_0 = I_q \cap [a, a + B_q], \qquad J_1 = I_q \cap (a + B_q, u_q].$$

Since both reflected sets are contained in I_q and have no points below a, their total length decomposes as

$$\begin{aligned} |E_+^{\text{large}}(r)| + |\widetilde{E}_-^{\text{large}}(r)| = &|E_+^{\text{large}}(r) \cap J_0| + |\widetilde{E}_-^{\text{large}}(r) \cap J_0| \\ &+ |E_+^{\text{large}}(r) \cap J_1| + |\widetilde{E}_-^{\text{large}}(r) \cap J_1|. \end{aligned}$$

On J_0, both sets may contribute, but each contribution is at most $|J_0|$, and

$$|J_0| \le \min\{B_q, |I_q|\}.$$

On J_1, the disjointness above gives

$$\big(E_+^{\mathrm{large}}(r) \cap J_1\big) \cap \big(\widetilde{E}_-^{\mathrm{large}}(r) \cap J_1\big) = \emptyset,$$

so

$$|E_+^{\mathrm{large}}(r) \cap J_1| + |\widetilde{E}_-^{\mathrm{large}}(r) \cap J_1| = |\big(E_+^{\mathrm{large}}(r) \cup \widetilde{E}_-^{\mathrm{large}}(r)\big) \cap J_1| \le |J_1|.$$

Since $I_q = [\max\{L_q, ru_q - K_q\}, u_q]$, the containment of $E_+^{\mathrm{large}}(r)$ and $\widetilde{E}_-^{\mathrm{large}}(r)$ in I_q gives $a \ge \max\{L_q, ru_q - K_q\}$. Hence

$$\begin{aligned}
|J_1| &= |I_q \cap (a + B_q, u_q]| \\
&\le (u_q - a - B_q)_+ \\
&\le (u_q - \max\{L_q, ru_q - K_q\} - B_q)_+ = (|I_q| - B_q)_+.
\end{aligned}$$

Therefore

$$\begin{aligned}
|E_+^{\mathrm{large}}(r)| + |E_-^{\mathrm{large}}(r)| &= |E_+^{\mathrm{large}}(r)| + |\widetilde{E}_-^{\mathrm{large}}(r)| \\
&\le 2|J_0| + |\big(E_+^{\mathrm{large}}(r) \cup \widetilde{E}_-^{\mathrm{large}}(r)\big) \cap J_1| \\
&\le 2\min\{B_q, |I_q|\} + (|I_q| - B_q)_+ \\
&\le B_q + |I_q| \\
&= B_q + u_q - \max\{L_q, ru_q - K_q\} \\
&\le (1 - r)u_q + K_q + B_q.
\end{aligned}$$

After decreasing q_0 if necessary, $1 - r^2/u_q \ge 1/2$. Thus (219) gives

$$B_q \le 2(2K_q + 2r^2) + 4\sqrt{u_q} \le C\{K_q + \sqrt{u_q} + 1\}.$$

Consequently, since $K_q = \sqrt{8 \log u_q}$,

$$K_q + B_q \le C\{K_q + \sqrt{u_q} + 1\} \le C\sqrt{u_q},$$

which proves (212). $\square$

The next lemma bounds $|E_\pm^{\mathrm{small}}(r)|$.

Lemma 4.6. *With $E_\pm^{\mathrm{small}}(r)$ defined in (211), there are universal constants $C < \infty$ and $q_0 > 0$ such that, whenever $q \le q_0$,*

$$|E_+^{\mathrm{small}}(r)| + |E_-^{\mathrm{small}}(r)| \le C(K_q + 1) \le C\sqrt{u_q} \tag{230}$$

uniformly over all positive measures Π on $\mathbb{R} \times [-1, 1]$ such that $\Pi(\mathbb{R} \times [-1, 1]) \le 1$, and over all families $(\mathcal{C}_z)$ satisfying (209).

Proof. Put

$$b = \frac{r}{u_q}, \qquad \eta = \Pi\{(\theta, \alpha) : |\alpha| < b\}.$$

For each z and u, put

$$Q_z(u) = \frac{G_z^{\mathrm{small}}(u)}{p(u)} = \frac{1}{p(u)} \int_{\{|\alpha| < b\}} \mathbb{P}\{|\theta + \alpha z + \sqrt{1 - \alpha^2}\, \varepsilon| \ge u\}\, \mathrm{d}\Pi(\theta, \alpha), \tag{231}$$

where $\varepsilon \sim N(0,1)$. If $z \in E_+^{\rm small}(r) \cup E_-^{\rm small}(r)$ and $u \in \mathcal{C}_z$ is one of the values appearing in (211), then

$$Q_z(u) \geq \frac{1}{4q}. \tag{232}$$

If $E_+^{\rm small}(r) \cup E_-^{\rm small}(r)$ is empty, there is nothing to prove. Otherwise choose $z_1 \in E_+^{\rm small}(r) \cup E_-^{\rm small}(r)$ so that $|z_1|$ is arbitrarily close to

$$\inf\{|z| :\ z \in E_+^{\rm small}(r) \cup E_-^{\rm small}(r)\}.$$

Choose $v_1 \in \mathcal{C}_{z_1}$ as in (211). Since $Q_{z_1}(v_1) \geq 1/(4q) > 0$, necessarily $\eta > 0$; if $\eta = 0$, then $Q_z \equiv 0$ for every z. By (211), $|z_1| \leq u_q$ and $||z_1| - rv_1| \leq K_q$. Therefore

$$rv_1 \leq |z_1| + K_q \leq u_q + K_q, \qquad \text{and hence} \qquad bv_1 \leq 1 + K_q/u_q. \tag{233}$$

Define

$$H_1 = \log\left\{\frac{Q_{z_1}(v_1)\sqrt{1-b^2}}{2\eta}\right\}.$$

Using (232) and $\eta \leq 1$,

$$H_1 \geq \log\left\{\frac{\sqrt{1-b^2}}{8q}\right\}.$$

For q_0 small enough, $u_q \geq 2$. Since $0 \leq r \leq 1$, this gives

$$b = \frac{r}{u_q} \leq \frac{1}{2}, \qquad \sqrt{1-b^2} \geq \frac{1}{2}.$$

Therefore, using $q = p(u_q) \leq e^{-u_q^2/2}$ from (299),

$$H_1 \geq \log\frac{1}{16q} \geq \frac{u_q^2}{2} - \log 16 \geq \frac{u_q^2}{3} \geq 4, \tag{234}$$

after decreasing q_0, if necessary. Moreover, by (233),

$$\frac{9b^2v_1^2}{1-b^2} \leq \frac{9(1+K_q/u_q)^2}{1-u_q^{-2}} \leq C.$$

Since $H_1 \geq u_q^2/3$, decreasing q_0 once more makes this last constant at most H_1. Hence

$$H_1 = \log\left\{\frac{Q_{z_1}(v_1)\sqrt{1-b^2}}{2\eta}\right\} \geq 4, \qquad \frac{9b^2v_1^2}{1-b^2} \leq H_1. \tag{235}$$

We will show the following result at the end of the proof:

$$\exists\, v \geq v_1 : \qquad Q_{z_1}(v) \geq \frac{3e^{10u_q}}{q}, \qquad 0 \leq v - v_1 \leq C\frac{v_1}{u_q}. \tag{236}$$

Now we prove (230) under (236). For any $|z'| \leq u_q$, put

$$\Delta = b|z' - z_1| \leq 2r. \tag{237}$$

Fix a component (θ, α) with $|\alpha| < b$, and use the same residual ε to form the two conditional statistics at z_1 and at z':

$$Y_{z_1} = \theta + \alpha z_1 + \sqrt{1-\alpha^2}\,\varepsilon, \qquad Y_{z'} = \theta + \alpha z' + \sqrt{1-\alpha^2}\,\varepsilon.$$

Then

$$|Y_{z'} - Y_{z_1}| = |\alpha|\,|z' - z_1| \leq b|z' - z_1| = \Delta,$$

and hence

$$\{|Y_{z_1}| \geq v\} \subseteq \{|Y_{z'}| \geq v - \Delta\}.$$

For this fixed (θ, α), the inclusion gives

$$\mathbb{P}_\varepsilon\{|Y_{z'}| \geq v - \Delta\} \geq \mathbb{P}_\varepsilon\{|Y_{z_1}| \geq v\}.$$

Integrating the last inequality with respect to Π over the small-coefficient set $\{|\alpha| < b\}$ gives, by the definition of G_z^{small},

$$\begin{aligned} G_{z'}^{\mathrm{small}}(v - \Delta) &= \int_{\{|\alpha|<b\}} \mathbb{P}_\varepsilon\{|Y_{z'}| \geq v - \Delta\}\ \mathrm{d}\Pi(\theta, \alpha) \\ &\geq \int_{\{|\alpha|<b\}} \mathbb{P}_\varepsilon\{|Y_{z_1}| \geq v\}\ \mathrm{d}\Pi(\theta, \alpha) = G_{z_1}^{\mathrm{small}}(v). \end{aligned}$$

Dividing by $p(v - \Delta)$ and using $G_{z_1}^{\mathrm{small}}(v) = Q_{z_1}(v)p(v)$ gives

$$Q_{z'}(v - \Delta) \geq Q_{z_1}(v)\frac{p(v)}{p(v-\Delta)}. \tag{238}$$

Since $p = 2\overline{\Phi}$, $p'(t)/p(t) = -\lambda(t)$, where $\lambda(t) = \phi(t)/\overline{\Phi}(t)$. Also, by (237), $r \leq 3/5$, and $v \geq v_1 \geq u_q$,

$$v - \Delta \geq v_1 - \Delta \geq u_q - 2r \geq u_q - \frac{6}{5} > 0$$

after decreasing q_0, if necessary. Thus the interval $[v - \Delta, v]$ lies in $(0, \infty)$, and

$$\frac{p(v)}{p(v-\Delta)} = \exp\left\{-\int_{v-\Delta}^{v} \lambda(t)\ \mathrm{d}t\right\}.$$

By (293) of Lemma A.2, for $v - \Delta \leq t \leq v$,

$$\lambda(t) \leq t + \frac{1}{t} \leq v + \frac{1}{v - \Delta}.$$

Therefore

$$\frac{p(v)}{p(v-\Delta)} \geq \exp\left[-\Delta\left\{v + \frac{1}{v-\Delta}\right\}\right]. \tag{239}$$

By (211),

$$rv_1 \leq |z_1| + K_q \leq u_q + K_q.$$

Also, (236) and (233) give

$$r(v - v_1) \leq C\frac{rv_1}{u_q} \leq C\left(1 + \frac{K_q}{u_q}\right) = O(1).$$

Therefore

$$rv = rv_1 + r(v - v_1) \le u_q + K_q + O(1).$$

Since $\Delta \le 2r$ and $v - \Delta \ge u_q - 2r$, the exponent in (239) satisfies

$$\Delta\left\{v + \frac{1}{v-\Delta}\right\} \le 2rv + \frac{2r}{u_q - 2r} \le 2u_q + 2K_q + O(1) \le 3u_q,$$

after decreasing q_0, if necessary, where the last step uses $K_q = o(u_q)$ from (201). Combining (238), (236), and (239),

$$\begin{aligned} Q_{z'}(v-\Delta) &\ge Q_{z_1}(v)\frac{p(v)}{p(v-\Delta)} \\ &\ge \frac{3e^{10u_q}}{q}\exp\left[-\Delta\left\{v + \frac{1}{v-\Delta}\right\}\right] \\ &\ge \frac{3e^{10u_q}}{q}e^{-3u_q} = \frac{3e^{7u_q}}{q} > \frac{3}{q}. \end{aligned}$$

Hence

$$Q_{z'}(v-\Delta) > \frac{3}{q}. \tag{240}$$

If $v - \Delta < u_q$, then

$$v_1 \le v < u_q + \Delta \le u_q + 2r. \tag{241}$$

Since $r \le 3/5$, (297) in Lemma A.2 gives, uniformly over $0 \le h \le 2r$,

$$\begin{aligned} \log\frac{p(u_q+h)}{p(u_q)} &= -u_q h - \frac{h^2}{2} + \log\frac{u_q}{u_q+h} + o(1) \\ &\ge -2ru_q - 2r^2 - \log\left(1 + \frac{2r}{u_q}\right) + o(1) \\ &\ge -\frac{6}{5}u_q - \frac{18}{25} - \log\left(1 + \frac{6}{5u_q}\right) + o(1) \\ &\ge -2u_q, \end{aligned}$$

after decreasing q_0, if necessary. The last step uses $u_q \to \infty$ as $q \downarrow 0$, and the $o(1)$ term is uniform over $0 \le h \le 2r \le 6/5$. Taking $h = 2r$ and using (241) together with the monotonicity of p,

$$p(v) \ge p(u_q + 2r) \ge p(u_q)e^{-2u_q} = qe^{-2u_q}.$$

Because $G_{z_1}^{\rm small}(v) \le \Pi(\mathbb{R}\times[-1,1]) \le 1$,

$$Q_{z_1}(v) = \frac{G_{z_1}^{\rm small}(v)}{p(v)} \le \frac{1}{p(v)} \le \frac{e^{2u_q}}{q}.$$

This contradicts (236), which gives $Q_{z_1}(v) \ge 3e^{10u_q}/q$. Hence

$$v - \Delta \ge u_q. \tag{242}$$

We next show

$$\mathcal{C}_{z'} \cap (v-\Delta, \infty) = \emptyset. \tag{243}$$

Suppose, to the contrary, that $u' \in \mathcal{C}_{z'}$ and $u' > v - \Delta$. Since (242) holds, the defining upper-tail condition in (209), applied with $w = v - \Delta < u'$, gives

$$G_{z'}(v-\Delta) \le 3\frac{p(v-\Delta)}{q}.$$

On the other hand, (240) gives

$$G_{z'}^{\mathrm{small}}(v-\Delta) = p(v-\Delta)Q_{z'}(v-\Delta) > 3\frac{p(v-\Delta)}{q},$$

and $G_{z'}(v-\Delta) \ge G_{z'}^{\mathrm{small}}(v-\Delta)$. This is a contradiction. Therefore every $u' \in \mathcal{C}_{z'}$ satisfies

$$u' \le v - \Delta \le v. \tag{244}$$

Therefore, (244) is proved.

Now let $z' \in E_+^{\mathrm{small}}(r) \cup E_-^{\mathrm{small}}(r)$. By (211), there exists $u' \in \mathcal{C}_{z'}$ such that

$$||z'| - ru'| \le K_q.$$

Together with (244), this gives

$$|z'| \le ru' + K_q \le rv + K_q \le |z_1| + 2K_q + C. \tag{245}$$

Here the last inequality follows from

$$rv = rv_1 + r(v - v_1) \le |z_1| + K_q + C,$$

using $||z_1| - rv_1| \le K_q$ from (211) and the bound $r(v - v_1) \le C$ proved above.

Let

$$a = \inf\{|z| : \ z \in E_+^{\mathrm{small}}(r) \cup E_-^{\mathrm{small}}(r)\}.$$

The point z_1 was chosen with $|z_1|$ arbitrarily close to a. Thus (245) shows that

$$\{|z| : \ z \in E_+^{\mathrm{small}}(r) \cup E_-^{\mathrm{small}}(r)\} \subseteq [a, a + 2K_q + C].$$

By (211),

$$E_+^{\mathrm{small}}(r) \subseteq [L_q, u_q], \qquad E_-^{\mathrm{small}}(r) \subseteq [-u_q, -L_q].$$

On $E_+^{\mathrm{small}}(r)$, the map $z \mapsto |z|$ is the identity. On $E_-^{\mathrm{small}}(r)$, it is the reflection $z \mapsto -z$. Thus,

$$|E_\pm^{\mathrm{small}}(r)| = \big|\{|z| : \ z \in E_\pm^{\mathrm{small}}(r)\}\big|.$$

As a result,

$$|E_+^{\mathrm{small}}(r)| + |E_-^{\mathrm{small}}(r)| \le 2(2K_q + C) \le C(K_q + 1).$$

This proves (230).

Proof of (236)**.** Put

$$R = 12e^{10u_q}, \qquad H_1 = \log\left\{\frac{Q_{z_1}(v_1)\sqrt{1-b^2}}{2\eta}\right\}.$$

The choice of v_1 in (211), together with (232), gives

$$Q_{z_1}(v_1) \geq \frac{1}{4q}.$$

Also, (234) gives $H_1 \geq u_q^2/3$. Thus it is enough to find $v \geq v_1$ such that

$$Q_{z_1}(v) \geq RQ_{z_1}(v_1), \qquad v - v_1 \leq C\frac{v_1}{u_q}.$$

For $z \in \mathbb{R}$ and $x \geq 0$, define

$$\kappa_z(x) = \int_{\{|\alpha|<b\}} \frac{\phi\left(\frac{x-(\theta+\alpha z)}{\sqrt{1-\alpha^2}}\right) + \phi\left(\frac{x+(\theta+\alpha z)}{\sqrt{1-\alpha^2}}\right)}{2\phi(x)\sqrt{1-\alpha^2}} \, \mathrm{d}\Pi(\theta,\alpha).$$

By definition of Q_z in (231),

$$Q_{z_1}(u) = \frac{\int_u^\infty 2\phi(x)\kappa_{z_1}(x) \, \mathrm{d}x}{\int_u^\infty 2\phi(x) \, \mathrm{d}x}.$$

Using straightforward algebra,

$$Q_{z_1}'(x) = \lambda(x)\{Q_{z_1}(x) - \kappa_{z_1}(x)\}, \qquad \lambda(x) = \phi(x)/\overline{\Phi}(x).$$

Define

$$\tilde{v} = \inf\left\{x \geq v_1 : \ Q_{z_1}(x) \geq RQ_{z_1}(v_1) \text{ or } \kappa_{z_1}(x) \geq \frac{1}{2}Q_{z_1}(x)\right\}. \tag{246}$$

Before $\tilde{v}$, we have $\kappa_{z_1}(x) < Q_{z_1}(x)/2$, and therefore

$$Q_{z_1}'(x) \geq \frac{1}{2}\lambda(x)Q_{z_1}(x) \geq \frac{x}{2}Q_{z_1}(x) \geq \frac{v_1}{2}Q_{z_1}(x).$$

Thus

$$(\log Q_{z_1})'(x) = \frac{Q_{z_1}'(x)}{Q_{z_1}(x)} \geq \frac{v_1}{2}.$$

As a result,

$$Q_{z_1}\left(v_1 + \frac{2\log R}{v_1}\right) \geq Q_{z_1}(v_1)\exp\left\{\frac{v_1}{2}\frac{2\log R}{v_1}\right\} = RQ_{z_1}(v_1),$$

and hence

$$\tilde{v} \leq v_1 + \frac{2\log R}{v_1}. \tag{247}$$

Now we consider two cases depending on which inequality is binding in (246). If $Q_{z_1}(\tilde{v}) \geq RQ_{z_1}(v_1)$, take $v = \tilde{v}$. Since $v_1 \geq u_q$ by (209),

$$\frac{\log R}{v_1} \leq C\frac{u_q}{v_1} \leq C\frac{v_1}{u_q},$$

Together with (247), this gives

$$\tilde{v} \leq v_1 + C\frac{u_q}{v_1} \leq v_1\left(1 + \frac{C}{u_q}\right). \tag{248}$$

Then (236) holds due to (248).

It remains to consider the case $\kappa_{z_1}(\tilde{v}) \ge Q_{z_1}(\tilde{v})/2$. Since Q_{z_1} is increasing on $[v_1, \tilde{v}]$, this gives $\kappa_{z_1}(\tilde{v}) \ge Q_{z_1}(v_1)/2$. We will show that (236) is satisfied with

$$v = \tilde{v} + 72\frac{\tilde{v}}{u_q}. \tag{249}$$

For $z \in \mathbb{R}$ and $x \ge 0$, define

$$\widetilde{\kappa}_z(x) = \exp\left\{\frac{b^2x^2}{2(1-b^2)}\right\}\kappa_z(x). \tag{250}$$

For a fixed pair (θ, α) with $|\alpha| < b$, the corresponding integrand in $\kappa_{z_1}(x)$ is

$$\begin{aligned}
&\frac{1}{2\phi(x)\sqrt{1-\alpha^2}}\left\{\phi\left(\frac{x-(\theta+\alpha z_1)}{\sqrt{1-\alpha^2}}\right)+\phi\left(\frac{x+(\theta+\alpha z_1)}{\sqrt{1-\alpha^2}}\right)\right\}\\
&\quad=\frac{1}{2\sqrt{1-\alpha^2}}\left[\exp\left\{-\frac{\{x-(\theta+\alpha z_1)\}^2}{2(1-\alpha^2)}+\frac{x^2}{2}\right\}\right.\\
&\qquad\qquad\left.+\exp\left\{-\frac{\{x+(\theta+\alpha z_1)\}^2}{2(1-\alpha^2)}+\frac{x^2}{2}\right\}\right]\\
&\quad=\frac{1}{\sqrt{1-\alpha^2}}\exp\left\{-\frac{\alpha^2x^2}{2(1-\alpha^2)}-\frac{(\theta+\alpha z_1)^2}{2(1-\alpha^2)}\right\}\cosh\left(\frac{(\theta+\alpha z_1)x}{1-\alpha^2}\right).
\end{aligned}$$

Multiplying by the factor in (250), the corresponding integrand in $\widetilde{\kappa}_{z_1}(x)$ is denoted by $f_{\theta,\alpha}(x)$, where

$$\begin{aligned}
f_{\theta,\alpha}(x) = {}&\frac{1}{\sqrt{1-\alpha^2}}\exp\left\{-\frac{(\theta+\alpha z_1)^2}{2(1-\alpha^2)}\right\}\exp\left\{\left(\frac{b^2}{2(1-b^2)}-\frac{\alpha^2}{2(1-\alpha^2)}\right)x^2\right\}\\
&\times\cosh\left(\frac{(\theta+\alpha z_1)x}{1-\alpha^2}\right).
\end{aligned} \tag{251}$$

The first two factors are positive and independent of x. The coefficient of x^2 in the second term is

$$a = \frac{b^2}{2(1-b^2)} - \frac{\alpha^2}{2(1-\alpha^2)} = \frac{b^2-\alpha^2}{2(1-b^2)(1-\alpha^2)} \ge 0,$$

because $|\alpha| < b$. Thus, taking logarithms in (251) and dropping the additive constant independent of x, the logarithm of the integrand has the form

$$ax^2 + \log\cosh(cx), \qquad c = \frac{\theta+\alpha z_1}{1-\alpha^2}.$$

On $[0, \infty)$,

$$\begin{aligned}
\frac{d}{dx}\{ax^2+\log\cosh(cx)\} &= 2ax + c\tanh(cx)\\
&= 2ax + |c|\tanh(|c|x) \ge 0,\\
\frac{d^2}{dx^2}\{ax^2+\log\cosh(cx)\} &= 2a + c^2\operatorname{sech}^2(cx) \ge 0.
\end{aligned}$$

Thus $f_{\theta,\alpha}$ is log-convex.

To pass from each integrand to the mixture, fix $0 \le t \le 1$ and $x, y \ge 0$. Log-convexity gives

$$f_{\theta,\alpha}\{tx+(1-t)y\} \le f_{\theta,\alpha}(x)^t f_{\theta,\alpha}(y)^{1-t}.$$

After integrating over $\{|\alpha| < b\}$, Hölder's inequality gives

$$\begin{aligned}\widetilde{\kappa}_{z_1}\{tx + (1-t)y\} &\le \int_{\{|\alpha|<b\}} f_{\theta,\alpha}(x)^t f_{\theta,\alpha}(y)^{1-t} \, \mathrm{d}\Pi(\theta,\alpha) \\ &\le \left\{\int_{\{|\alpha|<b\}} f_{\theta,\alpha}(x) \, \mathrm{d}\Pi(\theta,\alpha)\right\}^t \left\{\int_{\{|\alpha|<b\}} f_{\theta,\alpha}(y) \, \mathrm{d}\Pi(\theta,\alpha)\right\}^{1-t} \\ &= \widetilde{\kappa}_{z_1}(x)^t \widetilde{\kappa}_{z_1}(y)^{1-t}.\end{aligned}$$

Hence $\widetilde{\kappa}_{z_1}$ is log-convex. Equivalently,

$$x \mapsto \log \kappa_{z_1}(x) + \frac{b^2 x^2}{2(1-b^2)} \quad \text{is convex on } [0,\infty). \tag{252}$$

Fix $M < \infty$ and restrict $0 \le x \le M$. From (251), for a constant $C_M < \infty$,

$$0 \le f_{\theta,\alpha}(x) \le C_M \exp\left\{-\frac{(\theta+\alpha z_1)^2}{2(1-\alpha^2)} + \frac{M|\theta+\alpha z_1|}{1-\alpha^2}\right\} \le C_M. \tag{253}$$

The last inequality follows from

$$-\frac{(\theta+\alpha z_1)^2}{2(1-\alpha^2)} + \frac{M|\theta+\alpha z_1|}{1-\alpha^2} = -\frac{\{|\theta+\alpha z_1| - M\}^2}{2(1-\alpha^2)} + \frac{M^2}{2(1-\alpha^2)} \le \frac{M^2}{2(1-b^2)}.$$

Differentiating the integrand gives

$$\frac{\mathrm{d}}{\mathrm{d}x} f_{\theta,\alpha}(x) = f_{\theta,\alpha}(x) \left[2\left\{\frac{b^2}{2(1-b^2)} - \frac{\alpha^2}{2(1-\alpha^2)}\right\} x + \frac{\theta+\alpha z_1}{1-\alpha^2} \tanh\left\{\frac{(\theta+\alpha z_1)x}{1-\alpha^2}\right\}\right].$$

The bracketed factor is controlled by $1 + |\theta+\alpha z_1|$. Indeed, on $0 \le x \le M$,

$$\left|2\left\{\frac{b^2}{2(1-b^2)} - \frac{\alpha^2}{2(1-\alpha^2)}\right\} x + \frac{\theta+\alpha z_1}{1-\alpha^2} \tanh\left\{\frac{(\theta+\alpha z_1)x}{1-\alpha^2}\right\}\right| \le C_M\{1 + |\theta+\alpha z_1|\},$$

because $|\tanh| \le 1$, $1-\alpha^2 \ge 1-b^2$, and $|\alpha| < b$. Combining this with (253) gives

$$\left|\frac{\mathrm{d}}{\mathrm{d}x} f_{\theta,\alpha}(x)\right| \le C_M\{1 + |\theta+\alpha z_1|\} \exp\left\{-\frac{(\theta+\alpha z_1)^2}{2(1-\alpha^2)} + \frac{M|\theta+\alpha z_1|}{1-\alpha^2}\right\}.$$

Moreover,

$$\begin{aligned}&\{1 + |\theta+\alpha z_1|\} \exp\left\{-\frac{(\theta+\alpha z_1)^2}{2(1-\alpha^2)} + \frac{M|\theta+\alpha z_1|}{1-\alpha^2}\right\} \\ &\quad = \{1 + |\theta+\alpha z_1|\} \exp\left\{-\frac{\{|\theta+\alpha z_1| - M\}^2}{2(1-\alpha^2)} + \frac{M^2}{2(1-\alpha^2)}\right\} \le C_M,\end{aligned}$$

uniformly in (θ,α) with $|\alpha| < b$, because $1-\alpha^2 \in [1-b^2, 1]$. Therefore

$$\sup_{0 \le x \le M} \left|\frac{\mathrm{d}}{\mathrm{d}x} f_{\theta,\alpha}(x)\right| \le C_M.$$

Thus, on each compact interval, both $f_{\theta,\alpha}$ and $\mathrm{d}f_{\theta,\alpha}/\,\mathrm{d}x$ are dominated by an integrable bound with respect to the finite measure Π. Hence $\widetilde{\kappa}_{z_1}$, and therefore κ_{z_1}, is differentiable on $[0,\infty)$.

When $x=0$,

$$\begin{aligned}\kappa_{z_1}(0) &= \int_{\{|\alpha|<b\}} \frac{\phi\left(\frac{\theta+\alpha z_1}{\sqrt{1-\alpha^2}}\right)}{\phi(0)\sqrt{1-\alpha^2}}\,\mathrm{d}\Pi(\theta,\alpha)\\ &\le \int_{\{|\alpha|<b\}} \frac{1}{\sqrt{1-\alpha^2}}\,\mathrm{d}\Pi(\theta,\alpha) \le \frac{\eta}{\sqrt{1-b^2}}.\end{aligned}$$

Together with $\kappa_{z_1}(\tilde{v}) \ge Q_{z_1}(v_1)/2$, this gives

$$\frac{\kappa_{z_1}(\tilde{v})}{\kappa_{z_1}(0)} \ge \frac{Q_{z_1}(v_1)\sqrt{1-b^2}}{2\eta} = e^{H_1},$$

by the definition of H_1. By (252), the derivative at $\tilde{v}$ is at least the secant slope from 0 to $\tilde{v}$:

$$\begin{aligned}\frac{d}{dx}\left\{\log\kappa_{z_1}(x) + \frac{b^2x^2}{2(1-b^2)}\right\}\bigg|_{x=\tilde{v}} &\ge \frac{\log\kappa_{z_1}(\tilde{v}) - \log\kappa_{z_1}(0) + b^2\tilde{v}^2/\{2(1-b^2)\}}{\tilde{v}}\\ &\ge \frac{H_1}{\tilde{v}}.\end{aligned}$$

The derivative of the convex function in (252) is nondecreasing.

By (248), (233), and (234), we have $\tilde{v} \le v_1 + Cu_q/v_1$, $bv_1 \le 1 + K_q/u_q$, and $H_1 \ge u_q^2/3$. Since $v_1 \ge u_q$, the first two bounds imply

$$b\tilde{v} \le bv_1\left(1+\frac{C}{u_q}\right) \le \left(1+\frac{K_q}{u_q}\right)\left(1+\frac{C}{u_q}\right) \le 2$$

after decreasing q_0, if necessary. Also, if $\tilde{v} \le x \le \tilde{v} + 72\tilde{v}/u_q + 1$, then $x \le 2\tilde{v}$. Therefore, using $b\tilde{v} \le 2$ and $b \le u_q^{-1}$,

$$\frac{b^2x\tilde{v}}{1-b^2} \le \frac{2b^2\tilde{v}^2}{1-b^2} \le C.$$

Because $H_1 \ge u_q^2/3$, decreasing q_0 once more makes this constant at most $H_1/2$. Thus,

$$\frac{b^2x\tilde{v}}{1-b^2} \le \frac{H_1}{2} \qquad \left(\tilde{v} \le x \le \tilde{v} + 72\frac{\tilde{v}}{u_q} + 1\right).$$

Hence, on this interval,

$$(\log\kappa_{z_1})'(x) \ge \frac{H_1}{\tilde{v}} - \frac{b^2x}{1-b^2} \ge \frac{H_1}{2\tilde{v}}. \tag{254}$$

For v defined in (249),

$$\begin{aligned}\log\frac{\kappa_{z_1}(v)}{\kappa_{z_1}(\tilde{v})} &= \int_{\tilde{v}}^{v} (\log\kappa_{z_1})'(x)\,\mathrm{d}x \ge \int_{\tilde{v}}^{v} \frac{H_1}{2\tilde{v}}\,\mathrm{d}x\\ &= \frac{H_1}{2\tilde{v}}(v-\tilde{v}) = \frac{H_1}{2\tilde{v}}\,72\frac{\tilde{v}}{u_q} = \frac{36H_1}{u_q}.\end{aligned}$$

Thus,

$$\kappa_{z_1}(v) \ge \kappa_{z_1}(\tilde{v})\exp\left\{\frac{36H_1}{u_q}\right\} \ge \frac{Q_{z_1}(v_1)}{2}\exp\left\{\frac{36H_1}{u_q}\right\}. \tag{255}$$

Since $H_1 \geq u_q^2/3$, after decreasing q_0, if necessary,

$$\frac{1-e^{-1}}{2} \exp\left\{\frac{36H_1}{u_q}\right\} \geq \frac{1-e^{-1}}{2} e^{12u_q} \geq 12e^{10u_q} = R. \tag{256}$$

Also, $v \geq \tilde{v} \geq v_1 \geq u_q$, so $v^{-1} \leq 1$ after decreasing q_0, if necessary. By (249),

$$[v, v+v^{-1}] \subseteq \left[\tilde{v}, \tilde{v} + 72\frac{\tilde{v}}{u_q} + 1\right].$$

The lower bound (254) therefore gives

$$(\log \kappa_{z_1})'(x) \geq 0, \qquad v \leq x \leq v + v^{-1},$$

and hence κ_{z_1} is nondecreasing on $[v, v+v^{-1}]$. By the definition of Q_{z_1},

$$\begin{aligned} Q_{z_1}(v) &\geq \frac{\int_v^{v+v^{-1}} 2\phi(x)\kappa_{z_1}(x)\,\mathrm{d}x}{2\overline{\Phi}(v)} \\ &\geq \kappa_{z_1}(v)\frac{\overline{\Phi}(v) - \overline{\Phi}(v+v^{-1})}{\overline{\Phi}(v)} \end{aligned}$$

Moreover, by (293) of Lemma A.2,

$$\frac{\overline{\Phi}(v+v^{-1})}{\overline{\Phi}(v)} = \exp\left\{-\int_v^{v+v^{-1}} \lambda(t)\,\mathrm{d}t\right\} \leq \exp\left\{-\int_v^{v+v^{-1}} t\,\mathrm{d}t\right\} \leq e^{-1}. \tag{257}$$

Therefore

$$Q_{z_1}(v) \geq (1-e^{-1})\kappa_{z_1}(v) \geq RQ_{z_1}(v_1),$$

where the first inequality uses (257), and the second uses (255) and (256). Finally, by (249) and (248),

$$\begin{aligned} v - v_1 &= (\tilde{v} - v_1) + 72\frac{\tilde{v}}{u_q} \\ &\leq C\frac{u_q}{v_1} + 72\frac{v_1}{u_q}\left(1 + \frac{C}{u_q}\right). \end{aligned}$$

Since $v_1 \geq u_q$ from $v_1 \in \mathcal{C}_{z_1}$ and (209),

$$\frac{u_q}{v_1} \leq \frac{v_1}{u_q}, \qquad 72\frac{v_1}{u_q}\left(1+\frac{C}{u_q}\right) \leq C\frac{v_1}{u_q},$$

after increasing C and decreasing q_0, if necessary. Therefore

$$v - v_1 \leq C\frac{v_1}{u_q}.$$

Thus v defined in (249) satisfies (236). □

The following lemma is the deterministic input needed for the good-event part of the finite BH proof.

Lemma 4.7. *There are universal constants $C < \infty$ and $q_0 > 0$ with the following property. Let Π, ε, and G_z be as in* (208)*. Let $m \geq 1$ and $J \geq 2$. Define*

$$t_k = \frac{qk}{m}, \qquad u_k = p^{-1}(t_k). \tag{258}$$

For each z, let $w_k(z)$, $J < k \leq m$, be nonnegative measurable weights satisfying

$$\sum_{k>J} w_k(z) \leq 1.$$

and

$$G_z(u_k) \geq \frac{k}{2m}, \qquad G_z(u_j) \leq \frac{2j}{m} \quad (j > k) \text{ if } w_k(z) > 0. \tag{259}$$

Then, whenever $q \leq q_0$, uniformly over m, J, Π, the weights $(w_k(z))$, and $0 \leq r \leq 1$,

$$\int_{-u_q}^{u_q} \phi(z) \sum_{k>J} w_k(z) \rho_r(u_k, z)\, \mathrm{d}z \leq \frac{u_q}{2\sqrt{2\pi}} + C\sqrt{u_q}.$$

Proof of Lemma 4.7. For each z, put

$$\mathcal{C}_z = \{u_k : \ J < k \leq m, \ w_k(z) > 0\}.$$

If $u_k \in \mathcal{C}_z$, then the first condition in (259) gives

$$G_z(u_k) \geq \frac{k}{2m} = \frac{p(u_k)}{2q}. \tag{260}$$

For $u_q \leq v < u_k$, let $j = \lceil mp(v)/q \rceil$. Then $j > k$, $u_j \leq v$, and by monotonicity of G_z,

$$G_z(v) \leq G_z(u_j) \leq \frac{2j}{m} \leq 2\left(1 + \frac{1}{J}\right) \frac{p(v)}{q} \leq 3\frac{p(v)}{q},$$

where $J \geq 2$. Thus $\mathcal{C}_z$ satisfies (209). Also, since the weights are nonnegative and sum to at most one,

$$\sum_{k>J} w_k(z) \rho_r(u_k, z) \leq \sup_{u \in \mathcal{C}_z} \rho_r(u, z),$$

with the right side interpreted as zero when $\mathcal{C}_z$ is empty. It remains to prove

$$\int_{-u_q}^{u_q} \phi(z) \sup_{u \in \mathcal{C}_z} \rho_r(u, z)\, \mathrm{d}z \leq \frac{u_q}{2\sqrt{2\pi}} + C\sqrt{u_q}. \tag{261}$$

If $r = 1$, then Lemma 4.3 gives

$$\int_{-u_q}^{u_q} \phi(z) \sup_{u \in \mathcal{C}_z} \rho_1(u, z)\, \mathrm{d}z \leq \int_{-u_q}^{u_q} \phi(z) \sup_{u \geq u_q} \rho_1(u, z)\, \mathrm{d}z = 0.$$

This proves the desired bound in this case.

Next suppose $3/5 \leq r < 1$. Since $(1-r)/\sqrt{1-r^2} \leq 1/2$, Lemma 4.3 gives

$$\int_{-u_q}^{u_q} \phi(z) \sup_{u \geq u_q} \rho_r(u, z)\, \mathrm{d}z \leq \frac{1}{2} + \frac{u_q}{2\sqrt{2\pi}} \leq \frac{u_q}{2\sqrt{2\pi}} + C\sqrt{u_q}. \tag{262}$$

This upper-bounds the smaller supremum over $\mathcal{C}_z$ and proves the desired bound in this range.

It remains to consider $0 \le r < 3/5$. For $r = 0$, the integral is at most $1 - q$, which is bounded by the displayed right side after increasing C. Hence assume $0 < r < 3/5$. Use L_q, K_q defined in (201). When q is sufficiently small, (201) implies $L_q < u_q$. We will consider the following decomposition of (261):

$$\begin{aligned}
&\int_{-u_q}^{u_q} \phi(z) \sup_{u \in \mathcal{C}_z} \rho_r(u, z) \, \mathrm{d}z \\
&\le \int_{|z| < L_q} \phi(z) \sup_{u \in \mathcal{C}_z} \rho_r(u, z) \, \mathrm{d}z \\
&\quad + \int_{L_q \le |z| \le u_q} \phi(z) \sup_{\substack{u \in \mathcal{C}_z: \\ \left||z| - ru\right| > K_q}} \rho_r(u, z) \, \mathrm{d}z \\
&\quad + \int_{L_q \le |z| \le u_q} \phi(z) \sup_{\substack{u \in \mathcal{C}_z: \\ \left||z| - ru\right| \le K_q}} \rho_r(u, z) \, \mathrm{d}z.
\end{aligned} \tag{263}$$

All empty suprema in (263) are interpreted as zero.

By construction, every $u \in \mathcal{C}_z$ satisfies $u \ge u_q$. When $|z| < L_q$, Lemma 4.4 gives

$$\phi(z)\rho_r(u, z) \le C \left[e^{-(z - ru)^2/\{2(1 - r^2)\}} + e^{-(z + ru)^2/\{2(1 - r^2)\}} \right] \le C$$

uniformly over $u \in \mathcal{C}_z$. Taking the supremum over $\mathcal{C}_z$, with value zero when $\mathcal{C}_z$ is empty, and then integrating over a set of length at most $2L_q$ gives

$$\int_{|z| < L_q} \phi(z) \sup_{u \in \mathcal{C}_z} \rho_r(u, z) \, \mathrm{d}z \le C L_q. \tag{264}$$

On $L_q \le |z| \le u_q$, if $u \in \mathcal{C}_z$ and $\left||z| - ru\right| > K_q$, then

$$\min\{|z - ru|, |z + ru|\} = \left||z| - ru\right| > K_q.$$

Therefore (202) in Lemma 4.4 gives

$$\phi(z)\rho_r(u, z) \le C \exp\left\{ -\frac{K_q^2}{2(1 - r^2)} \right\} \le C e^{-K_q^2/2}.$$

Taking the supremum over such u, with value zero if there is no such u, and integrating over a set of length at most $2u_q$, gives

$$\int_{L_q \le |z| \le u_q} \phi(z) \sup_{\substack{u \in \mathcal{C}_z: \\ \left||z| - ru\right| > K_q}} \rho_r(u, z) \, \mathrm{d}z \le C u_q e^{-K_q^2/2} = O(u_q^{-3}). \tag{265}$$

where we use the definition $K_q = \sqrt{8 \log u_q}$ in (201).

Now consider the remaining term in (263). For any contributing pair, $|z| \in [L_q, u_q]$ and $\left||z| - ru\right| \le K_q$. Since $u \in \mathcal{C}_z$,

$$G_z^{\mathrm{large}}(u) + G_z^{\mathrm{small}}(u) = G_z(u) \ge \frac{p(u)}{2q}.$$

Thus

$$G_z^{\mathrm{large}}(u) \ge \frac{p(u)}{4q} \quad \text{or} \quad G_z^{\mathrm{small}}(u) \ge \frac{p(u)}{4q}.$$

Thus,

$$z \in E^{\text{large}}_{\text{sgn}(z)}(r) \cup E^{\text{small}}_{\text{sgn}(z)}(r),$$

and the last integral in (263) satisfies

$$\begin{aligned}
&\int_{L_q \le |z| \le u_q} \phi(z) \sup_{\substack{u \in \mathcal{C}_z: \\ \left||z| - ru\right| \le K_q}} \rho_r(u, z)\, \mathrm{d}z \\
&\le \int_{z \in E^{\text{large}}_{+}(r) \cup E^{\text{large}}_{-}(r)} \phi(z) \sup_{\substack{u \in \mathcal{C}_z: \\ \left||z| - ru\right| \le K_q}} \rho_r(u, z)\, \mathrm{d}z \\
&\quad + \int_{z \in E^{\text{small}}_{+}(r) \cup E^{\text{small}}_{-}(r)} \phi(z) \sup_{\substack{u \in \mathcal{C}_z: \\ \left||z| - ru\right| \le K_q}} \rho_r(u, z)\, \mathrm{d}z.
\end{aligned}$$

Lemma 4.5 gives

$$|E^{\text{large}}_{+}(r)| + |E^{\text{large}}_{-}(r)| \le (1 - r)u_q + C\sqrt{u_q}.$$

Lemma 4.6 gives

$$|E^{\text{small}}_{+}(r)| + |E^{\text{small}}_{-}(r)| \le C(K_q + 1) \le C\sqrt{u_q}.$$

Combining the two bounds, the total combined length of the corresponding aligned factor values is at most

$$(1 - r)u_q + C\sqrt{u_q}. \tag{266}$$

Combining (203) in Lemma 4.4 and (266), and using $0 < r < 3/5$, hence $(1 - r^2)^{-1/2} \le 5/4$, we obtain

$$\begin{aligned}
&\int_{L_q \le |z| \le u_q} \phi(z) \sup_{\substack{u \in \mathcal{C}_z: \\ \left||z| - ru\right| \le K_q}} \rho_r(u, z)\, \mathrm{d}z \\
&\le \left\{ \frac{1 + C(K_q + 1)/u_q}{2\sqrt{2\pi}\sqrt{1 - r^2}} + Ce^{-u_q} \right\} \{(1 - r)u_q + C\sqrt{u_q}\} \\
&\le \frac{1 - r}{2\sqrt{1 - r^2}\sqrt{2\pi}} u_q + C\frac{K_q + 1 + \sqrt{u_q}}{\sqrt{1 - r^2}} + Cu_q e^{-u_q} \\
&\le \frac{1 - r}{2\sqrt{1 - r^2}\sqrt{2\pi}} u_q + C\{K_q + 1 + \sqrt{u_q} + u_q e^{-u_q}\}.
\end{aligned}$$

Here the middle line expands the product and absorbs the term $(K_q + 1)/\sqrt{u_q}$ into $K_q + 1$. Since $L_q = K_q + \sqrt{u_q}$, $K_q = O(\sqrt{\log u_q})$, and $u_q e^{-u_q}$ is negligible, combining this with (263), (264), and (265) yields

$$\int_{-u_q}^{u_q} \phi(z) \sup_{u \in \mathcal{C}_z} \rho_r(u, z)\, \mathrm{d}z \le \frac{1 - r}{2\sqrt{1 - r^2}\sqrt{2\pi}} u_q + C\sqrt{u_q} \le \frac{u_q}{2\sqrt{2\pi}} + C\sqrt{u_q}, \tag{267}$$

where the last inequality uses $(1 - r)/\sqrt{1 - r^2} = \sqrt{(1 - r)/(1 + r)} \le 1$. This proves Lemma 4.7. □

The next lemma converts observed rejection counts into bounds for their conditional means.

Lemma 4.8. *There are universal constants $C, c > 0$ with the following property. Let $J \geq 1$, and let N_j be Poisson–binomial with mean μ_j; the counts may be nested in j. For $j \geq J$, define the bad events*

$$\mathcal{B}_j^+ = \{N_j \geq j-1,\ \mu_j < j/2\}, \tag{268}$$

$$\mathcal{B}_j^- = \{N_j \leq j-2,\ \mu_j > 2j\}. \tag{269}$$

Then

$$\mathbb{P}\left(\bigcup_{j\geq J}(\mathcal{B}_j^+ \cup \mathcal{B}_j^-)\right) \leq Ce^{-cJ}.$$

The constants are universal and the bound is uniform in the number and success probabilities of the Bernoulli variables.

Proof. If $\mu_j < j/2$, the Chernoff upper-tail inequality bounds $\mathbb{P}(\mathcal{B}_j^+) = \mathbb{P}(N_j \geq j-1)$ by $C_0e^{-c_0 j}$; if $\mu_j \geq j/2$, then $\mathcal{B}_j^+$ is empty. Similarly, if $\mu_j > 2j$, the Chernoff lower-tail inequality bounds $\mathbb{P}(\mathcal{B}_j^-) = \mathbb{P}(N_j \leq j-2)$ by $C_0e^{-c_0 j}$, while if $\mu_j \leq 2j$, then $\mathcal{B}_j^-$ is empty. Enlarging C_0, if necessary, absorbs the finitely many small values of j. Sum these bounds over $j \geq J$. □

Proof of Theorem 4.2 (b). If $m_0 = 0$, then $\mathrm{FDR}(\mathrm{BH}_q) = 0$. Hence assume $m_0 > 0$. Let R_i be the BH rejection count after replacing the true-null p-value P_i by zero, as in the proof of Lemma 4.1. For every hypothesis ℓ, define

$$F_{\ell,z}(t) = \mathbb{P}(P_\ell \leq t \mid Z = z), \qquad t_k = \frac{qk}{m}, \qquad u_k = p^{-1}(t_k).$$

The leave-one-out identity (191) and the same conditional-independence argument give, for almost every z,

$$\begin{aligned}\mathrm{FDR}(z) &= \frac{q}{m}\sum_{i\in\mathcal{H}_0}\sum_{k=1}^{m}\frac{F_{i,z}(t_k)}{t_k}\mathbb{P}(R_i = k \mid Z = z)\\ &= \frac{q}{m}\sum_{i\in\mathcal{H}_0}\sum_{k=1}^{m}\rho_{|a_i|}(u_k, z)\mathbb{P}(R_i = k \mid Z = z). \end{aligned}\tag{270}$$

Thus $\mathrm{FDR} = \int_{\mathbb{R}} \mathrm{FDR}(z)\phi(z)\,\mathrm{d}z$. The second equality follows from (199). Let

$$A_q = \{|Z| > u_q\}.$$

Decompose

$$\mathrm{FDR} = \mathbb{E}\{\mathrm{FDR}(Z)\,\mathbf{1}_{A_q}\} + \mathbb{E}\{\mathrm{FDR}(Z)\,\mathbf{1}_{A_q^c}\}.$$

The first term is at most q:

$$\mathbb{E}\{\mathrm{FDR}(Z)\,\mathbf{1}_{A_q}\} \leq \mathbb{P}(A_q) = 2\overline{\Phi}(u_q) = p(u_q) = q,$$

because $0 \leq \mathrm{FDR}(Z) \leq 1$ and $u_q = p^{-1}(q)$. It remains to prove

$$\mathbb{E}\{\mathrm{FDR}(Z)\,\mathbf{1}_{A_q^c}\} \leq \pi_0 q\left\{\frac{u_q}{2\sqrt{2\pi}} + C\sqrt{u_q}\right\}. \tag{271}$$

Define

$$J = \lceil\sqrt{u_q}\rceil.$$

Then $J \geq 2$ when q is sufficiently small. We split the summands in (271) into two parts. It suffices to prove

$$\begin{aligned} &\int_{A_q^c} \frac{q}{m} \sum_{i\in\mathcal{H}_0} \sum_{1\leq k\leq J} \rho_{|a_i|}(u_k, z)\mathbb{P}(R_i = k \mid Z = z)\phi(z)\,\mathrm{d}z \\ &\qquad \leq \pi_0 qJ \leq \pi_0 q(\sqrt{u_q} + 1), \end{aligned} \tag{272}$$

and

$$\begin{aligned} &\int_{A_q^c} \frac{q}{m} \sum_{i\in\mathcal{H}_0} \sum_{k>J} \rho_{|a_i|}(u_k, z)\mathbb{P}(R_i = k \mid Z = z)\phi(z)\,\mathrm{d}z \\ &\qquad \leq \pi_0 q\left\{\frac{u_q}{2\sqrt{2\pi}} + C\sqrt{u_q}\right\}. \end{aligned} \tag{273}$$

We first prove (272). Its left-hand side is bounded directly from (270). Using (200) with $u = u_k$, $\mathbf{1}_{A_q^c} \leq 1$, and $\mathbb{P}(R_i = k \mid Z = z) \leq 1$, it is at most

$$\frac{q}{m} \sum_{i\in\mathcal{H}_0} \sum_{1\leq k\leq J} 1 = \pi_0 qJ \leq \pi_0 q(\sqrt{u_q} + 1).$$

We next prove (273). For each i and z, and for $1 \leq j \leq m$, let

$$N_{i,j} = \sum_{\ell\neq i} \mathbf{1}\{P_\ell \leq qj/m\}, \qquad \mu_{i,j}(z) = \mathbb{E}(N_{i,j} \mid Z = z). \tag{274}$$

Define the good event corresponding to (268)–(269) as follows. Let

$$\mathcal{B}_{i,j}^{+}(z) = \{N_{i,j} \geq j-1,\ \mu_{i,j}(z) < j/2\}, \qquad \mathcal{B}_{i,j}^{-}(z) = \{N_{i,j} \leq j-2,\ \mu_{i,j}(z) > 2j\},$$

and set

$$\mathcal{A}_{i,z} := \bigcap_{j=J}^{m} \left((\mathcal{B}_{i,j}^{+}(z))^c \cap (\mathcal{B}_{i,j}^{-}(z))^c\right). \tag{275}$$

We split the conditional probability in the left-hand side of (273) according to this event:

$$\mathbb{P}(R_i = k \mid Z = z) = \mathbb{P}\{R_i = k, \mathcal{A}_{i,z} \mid Z = z\} + \mathbb{P}\{R_i = k, \mathcal{A}_{i,z}^c \mid Z = z\}.$$

It is enough to prove the bad-event bound

$$\begin{aligned} &\int_{A_q^c} \frac{q}{m} \sum_{i\in\mathcal{H}_0} \sum_{k>J} \rho_{|a_i|}(u_k, z)\mathbb{P}\{R_i = k, \mathcal{A}_{i,z}^c \mid Z = z\}\phi(z)\,\mathrm{d}z \\ &\qquad \leq C\pi_0 q, \end{aligned} \tag{276}$$

and the good-event bound

$$\begin{aligned} &\int_{A_q^c} \frac{q}{m} \sum_{i\in\mathcal{H}_0} \sum_{k>J} \rho_{|a_i|}(u_k, z)\mathbb{P}\{R_i = k, \mathcal{A}_{i,z} \mid Z = z\}\phi(z)\,\mathrm{d}z \\ &\qquad \leq \pi_0 q\left\{\frac{u_q}{2\sqrt{2\pi}} + C\sqrt{u_q}\right\}. \end{aligned} \tag{277}$$

We first prove the bad-event bound (276). Set $\delta_J = Ce^{-cJ}$. Fix i and z, and condition on $Z = z$. The residuals of the hypotheses $\ell \neq i$ are then independent, so the indicators in (274) are independent Bernoulli random variables, with success probabilities depending on ℓ, j, and z. Hence each $N_{i,j}$ is Poisson–binomial with mean $\mu_{i,j}(z)$, and the family may be nested in j, exactly as allowed in Lemma 4.8. Moreover, $\mathcal{A}_{i,z}^c$ is the union over $J \le j \le m$ of the two events

$$\{N_{i,j} \ge j-1,\ \mu_{i,j}(z) < j/2\} \quad \text{and} \quad \{N_{i,j} \le j-2,\ \mu_{i,j}(z) > 2j\},$$

which are exactly the bad events in Lemma 4.8 with $N_j = N_{i,j}$ and $\mu_j = \mu_{i,j}(z)$. Thus the lemma gives, uniformly in z,

$$\mathbb{P}(\mathcal{A}_{i,z}^c \mid Z = z) \le \delta_J. \tag{278}$$

Using (278), and since $u_k \ge u_q$ for $k > J$, for each i and z,

$$\begin{aligned}
&\sum_{k>J} \rho_{|a_i|}(u_k, z)\mathbb{P}\{R_i = k, \mathcal{A}_{i,z}^c \mid Z = z\} \\
&\quad \le \sup_{u \ge u_q} \rho_{|a_i|}(u, z) \sum_{k>J} \mathbb{P}\{R_i = k, \mathcal{A}_{i,z}^c \mid Z = z\} \\
&\quad \le \sup_{u \ge u_q} \rho_{|a_i|}(u, z)\mathbb{P}(\mathcal{A}_{i,z}^c \mid Z = z) \le \delta_J \sup_{u \ge u_q} \rho_{|a_i|}(u, z).
\end{aligned}$$

Since $A_q^c = \{|z| \le u_q\}$, inserting this pointwise bound into the left-hand side of (276) gives

$$\begin{aligned}
&\int_{A_q^c} \frac{q}{m} \sum_{i \in \mathcal{H}_0} \sum_{k>J} \rho_{|a_i|}(u_k, z)\mathbb{P}\{R_i = k, \mathcal{A}_{i,z}^c \mid Z = z\}\phi(z)\,\mathrm{d}z \\
&\quad \le \frac{q}{m} \sum_{i \in \mathcal{H}_0} \delta_J \int_{-u_q}^{u_q} \phi(z) \sup_{u \ge u_q} \rho_{|a_i|}(u, z)\,\mathrm{d}z \\
&\quad \le \frac{q}{m} |\mathcal{H}_0| \delta_J C_q = \pi_0 q \delta_J C_q \le C\pi_0 q(1 + u_q)e^{-c\sqrt{u_q}} \le C\pi_0 q.
\end{aligned} \tag{279}$$

Here Lemma 4.3 gives

$$C_q := \sup_{0 \le r \le 1} \int_{-u_q}^{u_q} \phi(z) \sup_{u \ge u_q} \rho_r(u, z)\,\mathrm{d}z \le \frac{1-q}{2} + \frac{u_q}{\sqrt{2\pi}}.$$

Lastly, we prove the good-event bound (277). Here $\mathcal{A}_{i,z}$ is defined in (275). On $\{R_i = k\}$,

$$N_{i,k} = k - 1, \qquad N_{i,j} \le j - 2 \quad (j > k). \tag{280}$$

Consequently, on $\{R_i = k\} \cap \mathcal{A}_{i,z}$, the exclusions in (275) applied with $j = k$ and then with each $j > k$ give

$$\mu_{i,k}(z) \ge k/2, \qquad \mu_{i,j}(z) \le 2j \quad (j > k). \tag{281}$$

Define the averaged conditional sub-CDF of the other hypotheses and its scale version by

$$F_{-i,z}(t) = \frac{1}{m} \sum_{\ell \ne i} F_{\ell,z}(t), \qquad G_{i,z}(u) = F_{-i,z}\{p(u)\}. \tag{282}$$

For each $\ell \ne i$, the common-factor representation (189) gives, conditionally on $Z = z$,

$$F_{\ell,z}\{p(u)\} = \mathbb{P}\{|T_\ell| \ge u \mid Z = z\} = \mathbb{P}\{|\theta_\ell + a_\ell z + \sqrt{1 - a_\ell^2}\,\varepsilon_\ell| \ge u\}.$$

Therefore

$$G_{i,z}(u) = \int \mathbb{P}\{|\theta + \alpha z + \sqrt{1 - \alpha^2}\,\varepsilon| \ge u\}\,\mathrm{d}\Pi_i(\theta, \alpha), \qquad \Pi_i = \frac{1}{m} \sum_{\ell \ne i} \delta_{(\theta_\ell, a_\ell)}.$$

This is exactly the form (208); the normalization by m matches the definition of $F_{-i,z}$, and Π_i has total mass $(m-1)/m \le 1$, as required there.

We now verify the hypotheses of Lemma 4.7, one true null i at a time. For this fixed i, define

$$w_{i,k}(z) = \mathbb{P}\{R_i = k, \mathcal{A}_{i,z} \mid Z = z\}, \qquad J < k \le m,$$

and take $G_z = G_{i,z}$. First, the weights are nonnegative and, because the events $\{R_i = k\}$, $1 \le k \le m$, are disjoint,

$$\sum_{k>J} w_{i,k}(z) \le 1.$$

Second, whenever $w_{i,k}(z) > 0$, the event $\{R_i = k, \mathcal{A}_{i,z}\}$ has positive conditional probability. Since the quantities $\mu_{i,j}(z)$ are fixed after conditioning on $Z = z$, the implication above gives (281) for this i, z, k. Using $\mu_{i,j}(z) = mG_{i,z}(u_j)$, these inequalities become

$$G_{i,z}(u_k) \ge \frac{k}{2m}, \qquad G_{i,z}(u_j) \le \frac{2j}{m} \quad (j > k).$$

These verify the rank-weight conditions for every k with $w_{i,k}(z) > 0$, which is the required support-restricted hypothesis in Lemma 4.7, with $r = |a_i|$. Therefore, for this fixed i,

$$\int_{-u_q}^{u_q} \sum_{k>J} w_{i,k}(z)\rho_{|a_i|}(u_k, z)\phi(z)\,\mathrm{d}z \le \frac{u_q}{2\sqrt{2\pi}} + C\sqrt{u_q},$$

where $A_q^c = \{|z| \le u_q\}$. Summing the preceding bound over $i \in \mathcal{H}_0$ and using $|\mathcal{H}_0|/m = \pi_0$ gives (277).

We have proved part (b) for $q \le q_0$. If $q_0 < q \le 2\bar{\Phi}(1)$, part (a) gives

$$\mathrm{FDR}(\mathrm{BH}_q) \le q\left\{1 + \pi_0\left(\frac{1-q}{2} + \frac{u_q}{\sqrt{2\pi}}\right)\right\}.$$

On the compact interval $q \in [q_0, 2\bar{\Phi}(1)]$, the quantities u_q and $\log(1/q)$ are bounded above and $\log(1/q)$ is bounded away from zero. Hence, after enlarging the universal constant C,

$$\frac{1-q}{2} + \frac{u_q}{\sqrt{2\pi}} \le \frac{1}{2\sqrt{\pi}}\sqrt{\log(1/q)} + C\{\log(1/q)\}^{1/4},$$

which closes part (b) on the remaining q-range. □

Remark 4.9 (PRDN for the two-sided null p-values)*.* The theorem imposes no sign restriction on the loadings a_i, so the null correlations a_ia_j may be negative. Nevertheless, the two-sided null p-values satisfy PRDN. Let

$$T_0 = (T_i : i \in \mathcal{H}_0), \qquad a_0 = (a_i : i \in \mathcal{H}_0), \qquad \varepsilon_0 = (\varepsilon_i : i \in \mathcal{H}_0).$$

Thus T_0 is the m_0-dimensional subvector containing precisely the test statistics for the true nulls; the subscript 0 denotes restriction to $\mathcal{H}_0$, not an additional coordinate. Put

$$D_0 = \mathrm{diag}(1 - a_i^2 : i \in \mathcal{H}_0).$$

Since $\theta_i = 0$ for $i \in \mathcal{H}_0$,

$$T_0 = a_0 Z + \sqrt{D_0}\,\varepsilon_0.$$

Hence T_0 is centered Gaussian with covariance matrix

$$\Sigma_0 = D_0 + a_0 a_0^\top.$$

Assume first that $|a_i| < 1$ for every null coordinate. Let

$$B = \operatorname{diag}\{\operatorname{sign}(a_i) : i \in \mathcal{H}_0\}, \qquad r_0 = (|a_i| : i \in \mathcal{H}_0)^\top,$$

with $\operatorname{sign}(0) = 1$. Then

$$B\Sigma_0 B = D_0 + r_0 r_0^\top.$$

By the Sherman–Morrison formula,

$$(D_0 + r_0 r_0^\top)^{-1} = D_0^{-1} - \frac{D_0^{-1} r_0 r_0^\top D_0^{-1}}{1 + r_0^\top D_0^{-1} r_0}.$$

Thus every off-diagonal entry of $(D_0 + r_0 r_0^\top)^{-1}$ is nonpositive. The Karlin–Rinott absolute-value multinormal criterion for multivariate total positivity of order two (MTP_2) [Karlin and Rinott, 1981] therefore applies to BT_0, and $(|(BT_0)_i| : i \in \mathcal{H}_0)$ is MTP_2. Since $|(BT_0)_i| = |T_i|$, coordinatewise sign changes do not alter the absolute null statistics. Applying the same decreasing map $g(x) = p(x) = 2\overline{\Phi}(x)$ to every coordinate preserves MTP_2. Indeed, for $u, v \in (0,1)^{m_0}$, coordinatewise monotonicity of g^{-1} gives

$$g^{-1}(u \wedge v) = g^{-1}(u) \vee g^{-1}(v), \qquad g^{-1}(u \vee v) = g^{-1}(u) \wedge g^{-1}(v).$$

If f is the density of $(|(BT_0)_i| : i \in \mathcal{H}_0)$, the density after the transformation is

$$\widetilde{f}(u) = f\{g^{-1}(u)\} \prod_{i \in \mathcal{H}_0} |(g^{-1})'(u_i)|.$$

For each coordinate, $\{u_i \wedge v_i, u_i \vee v_i\} = \{u_i, v_i\}$; hence the two products of Jacobian factors are equal. Therefore the MTP_2 inequality for f implies

$$\begin{aligned}
\widetilde{f}(u \wedge v)\widetilde{f}(u \vee v) &= f\{g^{-1}(u) \vee g^{-1}(v)\} f\{g^{-1}(u) \wedge g^{-1}(v)\} \prod_{i \in \mathcal{H}_0} |(g^{-1})'(u_i)(g^{-1})'(v_i)| \\
&\geq f\{g^{-1}(u)\} f\{g^{-1}(v)\} \prod_{i \in \mathcal{H}_0} |(g^{-1})'(u_i)(g^{-1})'(v_i)| \\
&= \widetilde{f}(u)\widetilde{f}(v).
\end{aligned}$$

Thus the vector of two-sided null p-values is MTP_2, which implies PRDN.

If some $|a_i| = 1$, define $a_i^{(\eta)} = (1-\eta)a_i$ for all $i \in \mathcal{H}_0$, and let $\eta \downarrow 0$. The corresponding absolute null vectors converge weakly to $(|T_i| : i \in \mathcal{H}_0)$. Under the measure-theoretic definition of MTP_2, the class of MTP_2 probability measures is closed under weak convergence [Colangelo et al., 2006]. Hence the limiting absolute-value law remains MTP_2. Appendix A.1 of Su [2018] then implies that the induced vector of two-sided null p-values satisfies PRDN.

4.3 Upper bounds for one-sided tests

We derive an FDR upper bound for one-sided tests under the common-factor model and show that its leading constant cannot be improved within this class.

Theorem 4.10. *Consider the common-factor model* (189)*. Test $H_i : \theta_i \le 0$ with $P_i = \overline{\Phi}(T_i)$. Let $\mathcal{H}_0 = \{i : \theta_i \le 0\}$, $m_0 = |\mathcal{H}_0|$, and $\pi_0 = m_0/m$. For $0 < q < 1/2$, put $u_q^+ = \overline{\Phi}^{-1}(q) > 0$. The FDR is zero when $m_0 = 0$, and otherwise*

$$\mathrm{FDR}(\mathrm{BH}_q) \le q\left\{2 + \pi_0\left(\frac{1}{2} + \frac{u_q^+}{\sqrt{2\pi}}\right)\right\}. \tag{283}$$

Consequently, the worst-case FDR in this common-factor class is $O\{q\sqrt{\log(1/q)}\}$, uniformly in m, the means $(\theta_1, \dots, \theta_m)$, and the loadings $(a_1, \dots, a_m)$, as $q \downarrow 0$.

The following corollary identifies the exact leading constant for the common-factor class.

Corollary 4.11. *For the common-factor model* (189)*, where $\mathrm{FDR}_{\theta,a}$ denotes the FDR under that model, as $q \downarrow 0$,*

$$\sup_{\substack{m\in\mathbb{N},\ \theta\in\mathbb{R}^m,\\ a\in[-1,1]^m}} \mathrm{FDR}_{\theta,a}(\mathrm{BH}_q) = \frac{q\sqrt{\log(1/q)}}{\sqrt{\pi}} + O(q).$$

Thus the constant $1/\sqrt{\pi}$ in Proposition 3.2 is optimal in the common-factor class.

Proof of Theorem 4.10. If there are no true nulls, the FDR is zero. Partition the true nulls into the two sign groups

$$\mathcal{H}_{0,1} = \{i \in \mathcal{H}_0 : a_i \ge 0\}, \qquad \mathcal{H}_{0,-1} = \{i \in \mathcal{H}_0 : a_i < 0\}.$$

Let $\mathcal{S} \subseteq \{-1, 1\}$ index the nonempty groups. The leave-one-out identity (191) and conditional reduction (192) are independent of the particular form of the p-values.

Suppose $i \in \mathcal{H}_{0,\sigma}$, where $\sigma \in \mathcal{S}$, and write $r = |a_i|$ and $Y_\sigma = \sigma Z \sim N(0,1)$. Define the actual conditional CDF by

$$F_{i,y}^{\theta_i}(t) = \mathbb{P}_{\theta_i}(P_i \le t \mid Y_\sigma = y)$$

and let $F_{r,y}^0$ denote the corresponding conditional CDF at the boundary mean $\theta_i = 0$. Conditional on $Y_\sigma = y$, $T_i \sim N(\theta_i + ry, 1 - r^2)$. Since $\theta_i \le 0$ and rejection is increasing in T_i,

$$F_{i,y}^{\theta_i}(t) \le F_{r,y}^0(t), \qquad 0 < t < 1.$$

Thus it suffices in the conditional leave-one-out bound to control the boundary CDF $F_{r,y}^0$.

For $r < 1$, the one-tail Gaussian density-ratio calculation gives

$$\phi(y) \sup_{0<t\le q} \frac{F_{r,y}^0(t)}{t} \le \frac{1}{\sqrt{1-r^2}\sqrt{2\pi}} \exp\left\{-\frac{(ru_q^+ - y)_+^2}{2(1-r^2)}\right\}. \tag{284}$$

Indeed, write $t = p_+(u)$, where $u \ge u_q^+$. The bivariate normal density identity

$$\phi(y)\frac{1}{\sqrt{1-r^2}}\phi\left(\frac{x - ry}{\sqrt{1-r^2}}\right) = \phi(x)\frac{1}{\sqrt{1-r^2}}\phi\left(\frac{y - rx}{\sqrt{1-r^2}}\right)$$

and $p_+(u) = \int_u^\infty \phi(x)\,\mathrm{d}x$ give

$$
\begin{aligned}
\phi(y)\frac{F^0_{r,y}\{p_+(u)\}}{p_+(u)} &= \frac{\displaystyle\int_u^\infty \phi(x)\frac{1}{\sqrt{1-r^2}}\phi\left(\frac{y-rx}{\sqrt{1-r^2}}\right)\,\mathrm{d}x}{\displaystyle\int_u^\infty \phi(x)\,\mathrm{d}x} \\
&\le \sup_{x\ge u}\frac{1}{\sqrt{1-r^2}}\phi\left(\frac{y-rx}{\sqrt{1-r^2}}\right) \\
&\le \frac{1}{\sqrt{1-r^2}\sqrt{2\pi}}\exp\left\{-\frac{(ru-y)_+^2}{2(1-r^2)}\right\}. \qquad (285)
\end{aligned}
$$

The final inequality in (285) follows because, for every $x \ge u$,

$$
(rx-y)^2 \ge (ru-y)_+^2.
$$

Since $r \ge 0$, the map $u \mapsto (ru-y)_+$ is nondecreasing. Thus the right-hand side of (285) is nonincreasing in u and is maximized over $u \ge u_q^+$ at $u = u_q^+$, proving (284). Integrating over $y \le u_q^+$ yields

$$
\begin{aligned}
I(r) &:= \int_{-\infty}^{u_q^+} \phi(y)\sup_{0<t\le q}\frac{F^0_{r,y}(t)}{t}\,\mathrm{d}y \\
&\le \frac{1}{\sqrt{1-r^2}\sqrt{2\pi}}\left\{\int_{-\infty}^{ru_q^+} e^{-(ru_q^+-y)^2/\{2(1-r^2)\}}\,\mathrm{d}y + \int_{ru_q^+}^{u_q^+} 1\,\mathrm{d}y\right\} \\
&= \frac{1}{\sqrt{1-r^2}\sqrt{2\pi}}\left\{\int_0^\infty e^{-v^2/\{2(1-r^2)\}}\,\mathrm{d}v + (1-r)u_q^+\right\} \\
&= \frac{1}{2} + \frac{(1-r)u_q^+}{\sqrt{1-r^2}\sqrt{2\pi}} \\
&\le \frac{1}{2} + \frac{u_q^+}{\sqrt{2\pi}}. \qquad (286)
\end{aligned}
$$

The inequality $(1-r)u_q^+/\{\sqrt{1-r^2}\sqrt{2\pi}\} \le u_q^+/\sqrt{2\pi}$ in (286) follows from $(1-r)/\sqrt{1-r^2} = \sqrt{(1-r)/(1+r)} \le 1$. If $r = 1$, then on $Y_\sigma < u_q^+$ the boundary-null p-value is larger than q, so (286) remains valid.

Let V_σ be the false rejections in sign group σ and $\mathrm{FDP}_\sigma = V_\sigma/(R \vee 1)$. On $A_\sigma = \{Y_\sigma \le u_q^+\}$, sum the per-null conditional bound over $\mathcal{H}_{0,\sigma}$ and apply (286). This gives

$$
\mathbb{E}(\mathrm{FDP}_\sigma \mathbf{1}_{A_\sigma}) \le q\frac{|\mathcal{H}_{0,\sigma}|}{m}\left(\frac{1}{2} + \frac{u_q^+}{\sqrt{2\pi}}\right).
$$

On A_σ^c, use $0 \le \mathrm{FDP}_\sigma \le 1$ and $\mathbb{P}(A_\sigma^c) = q$. Summing over $\sigma \in \mathcal{S}$, using $|\mathcal{S}| \le 2$ and $\mathrm{FDP} = \sum_{\sigma\in\mathcal{S}} \mathrm{FDP}_\sigma$, proves (283). □

Proof of Corollary 4.11. Theorem 3.1 gives the lower bound within the common-factor class. Indeed, for the perturbed model (32), set

$$
a_i^{(\eta)} = \sqrt{1-\eta^2}\,b_i, \qquad \tilde{\varepsilon}_i^{(\eta)} = \frac{\sqrt{(1-\eta^2)(1-b_i^2)}\,\varepsilon_i + \eta\xi_i}{\sqrt{1-\{a_i^{(\eta)}\}^2}}.
$$

The variables $\widetilde{\varepsilon}_i^{(\eta)}$ are independent standard normal variables and are independent of Z, and

$$T_i^{(\eta)} = \theta_i + a_i^{(\eta)} Z + \sqrt{1 - \{a_i^{(\eta)}\}^2}\, \widetilde{\varepsilon}_i^{(\eta)}.$$

Thus the perturbation is of the form (189). Together with Proposition 3.2, this gives

$$\sup_{\substack{m\in\mathbb{N},\ \theta\in\mathbb{R}^m,\\ a\in[-1,1]^m}} \mathrm{FDR}_{\theta,a}(\mathrm{BH}_q) \geq \ell_{\leq}(q) = \frac{q\sqrt{\log(1/q)}}{\sqrt{\pi}} + \frac{q}{2} + o(q).$$

For the matching upper bound, note first that $u_q^+ = p^{-1}(2q)$, so (298) in Lemma A.2 gives $u_q^+ \sim \sqrt{2\log(1/q)}$. More precisely, (295) in Lemma A.2 and $q = \overline{\Phi}(u_q^+)$ imply

$$(u_q^+)^2 = 2\log(1/q) - 2\log u_q^+ - \log(2\pi) + o(1) = 2\log(1/q) + O\{\log\log(1/q)\}.$$

Consequently,

$$u_q^+ = \sqrt{2\log(1/q)} + O\left\{\frac{\log\log(1/q)}{\sqrt{\log(1/q)}}\right\} = \sqrt{2\log(1/q)} + o(1).$$

Theorem 4.10, with $\pi_0 \leq 1$, now gives the reverse inequality up to an $O(q)$ term, proving the claim. $\square$

5 Discussion

For two-sided problems, Theorem 2.1 and Proposition 2.2 show that no bound of the form Cq, with C independent of q, can hold uniformly over all Gaussian correlation matrices. Because the lower-bound construction belongs to the common-factor class, Theorem 4.2 shows that the worst-case FDR within this class has the sharp leading constant $1/(2\sqrt{\pi})$ in the $q\sqrt{\log(1/q)}$ normalization. For one-sided problems, Theorem 3.1, Proposition 3.2, and Corollary 4.11 give the matching constant $1/\sqrt{\pi}$. Thus this constant is optimal in the common-factor class, with the upper bound allowing arbitrary means and scalar loadings.

There are also procedures with finite-sample FDR guarantees for dependent two-sided Gaussian tests. The conditional-calibration framework of Fithian and Lei [2022] decomposes FDR into hypothesis-wise contributions and, for each null i, conditions on a statistic under which the ith p-value remains conditionally valid and the relevant conditional law is tractable, often eliminating nuisance parameters. The resulting guarantees depend on the procedure and the dependence assumptions: under PRDS, dBH_1 controls FDR and contains BH, whereas under arbitrary dependence, dBY controls FDR and contains the Benjamini–Yekutieli procedure. Conditional calibration can therefore exploit known dependence without automatically paying the full harmonic-factor correction of Benjamini and Yekutieli [2001].

Another route is the fixed-X knockoff filter of Barber and Candès [2015]. In a Gaussian linear model, knockoffs augment the design with synthetic variables that mimic the correlation structure of the original variables, and the resulting original-versus-knockoff comparisons create null statistics with the symmetry needed for FDR control. Li and Fithian [2021] recast fixed-X knockoffs as adding user-generated Gaussian noise to a Gaussian estimator to obtain a whitened estimator, followed by conditional inference. This whitening can substantially reduce power when the covariance has large leading eigenvalues associated with dense leading eigenvectors, because it then requires enough added noise to erase much of the signal. Knockoffs also face a threshold phenomenon when the rejection set is small. The calibrated knockoff method of Luo et al. [2025] uses conditional calibration to uniformly improve knockoff procedures, with especially large gains in small-rejection regimes.

These findings point to two complementary directions: identifying structured dependence under which ordinary BH remains reliable, and developing procedures that use dependence information when the two-sided transformation destroys the monotonicity needed by standard positive-dependence arguments.

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A Auxiliary lemmas

This appendix collects the properties of the envelope and Gaussian tails used to define and analyze $\nu_{q,r}$.

The following lemma characterizes the likelihood-ratio envelope.

Lemma A.1. *For $0 \le y \le 1$, $M(y) = 1$. For $y > 1$, $a \mapsto L_a(y)$ has a unique maximizer $a(y) > 0$, characterized by*

$$a(y) = y \tanh\{a(y)y\}.$$

The maps $a : (1, \infty) \to (0, \infty)$ and $M : (1, \infty) \to (1, \infty)$ are continuous and strictly increasing, with

$$\lim_{y \downarrow 1} a(y) = 0, \quad \lim_{y \to \infty} M(y) = \infty. \tag{287}$$

The inverse $y = y(a)$ is continuously differentiable on $(0, \infty)$, and

$$y'(a) = \frac{1 - y(a)^2 \operatorname{sech}^2\{ay(a)\}}{\tanh\{ay(a)\} + ay(a) \operatorname{sech}^2\{ay(a)\}} > 0. \tag{288}$$

Moreover, with $a_q = a(y_q)$, the deficit in the secondary survival mass satisfies

$$1 - qM\{y(a)\} > 0 \quad (0 < a < a_q), \quad 1 - qM\{y(a_q)\} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}a}[1 - qM\{y(a)\}] < 0. \tag{289}$$

Proof. Let

$$g_y(a) = \log L_a(y) = -\frac{a^2}{2} + \log\cosh(ay).$$

Then

$$g_y'(a) = -a + y\tanh(ay). \tag{290}$$

If $y \le 1$, then $y\tanh(ay) \le ay^2 \le a$, so $M(y) = L_0(y) = 1$.

For $y > 1$, put $t = ay$. Equation $g_y'(a) = 0$ is equivalent to

$$y^2 = \frac{t}{\tanh t}. \tag{291}$$

To prove that the map $t \mapsto t/\tanh t$ in (291) is strictly increasing, let $H(t) = \tanh t - t\operatorname{sech}^2 t$. Then $H(0) = 0$ and $H'(t) = 2t\operatorname{sech}^2 t\tanh t > 0$ for $t > 0$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t}{\tanh t}\right) = \frac{\tanh t - t\operatorname{sech}^2 t}{\tanh^2 t} > 0.$$

Moreover, $t/\tanh t \to 1$ as $t \downarrow 0$ and $t/\tanh t \to \infty$ as $t \to \infty$. Because $y^2 > 1$, (291) in Lemma A.1 therefore has a unique positive solution. The sign of g_y' changes from positive to negative there, so the solution is the unique maximizer. Moreover,

$$a(y) = \sqrt{t\tanh t}, \quad y = \sqrt{\frac{t}{\tanh t}}, \tag{292}$$

Both parameter functions in (292) in Lemma A.1 are smooth and strictly increasing in $t > 0$. They tend respectively to zero and one as $t \downarrow 0$, and both tend to infinity as $t \to \infty$. Thus $a(y)$ is continuous and strictly increasing on $(1, \infty)$, with $a(y) \to 0$ as $y \downarrow 1$; the inverse function theorem applied to the second parameter function also shows that $a(y)$ is continuously differentiable. Since $M(y) = L_{a(y)}(y)$, the chain rule and $\partial_a L_{a(y)}(y) = 0$ give

$$M'(y) = M(y)a(y)\tanh\{a(y)y\} > 0.$$

Thus M is continuous and strictly increasing. Also

$$M(y) \ge L_y(y) \ge \frac{1}{2}e^{y^2/2} \to \infty.$$

Finally, apply the implicit function theorem to

$$F(a, y) = a - y\tanh(ay) = 0.$$

At the maximizer, with $t = ay > 0$, equation (291) in Lemma A.1 gives

$$y^2\operatorname{sech}^2(ay) = \frac{t\operatorname{sech}^2 t}{\tanh t} = \frac{2t}{\sinh(2t)} < 1.$$

Consequently,

$$F_a = 1 - y^2\operatorname{sech}^2(ay) > 0, \qquad F_y = -\tanh(ay) - ay\operatorname{sech}^2(ay) < 0.$$

Thus (288) in Lemma A.1 follows from $y' = -F_a/F_y$.

It remains to prove (289) in Lemma A.1. Since $a_q = a(y_q)$, we have $y(a_q) = y_q$, and therefore $1 - qM\{y(a_q)\} = 0$ by (4). If $0 < a < a_q$, then $y(a) < y_q$; since M is strictly increasing on $(1, \infty)$, $qM\{y(a)\} < qM(y_q) = 1$. Finally,

$$\frac{\mathrm{d}}{\mathrm{d}a}[1 - qM\{y(a)\}] = -qM'\{y(a)\}y'(a) < 0,$$

because $M' > 0$ on $(1, \infty)$ and $y' > 0$ on $(0, \infty)$. $\square$

The final lemma collects the Mills bounds and quantile asymptotics used to control Gaussian tails.

Lemma A.2. *The normal hazard λ is increasing on $\mathbb{R}$ and satisfies*

$$x < \lambda(x) < x + x^{-1} \quad (x > 0). \tag{293}$$

Moreover, as $x \to \infty$,

$$\lambda(x) = x + x^{-1} + O(x^{-3}). \tag{294}$$

As $x \to \infty$,

$$\overline{\Phi}(x) = \frac{\phi(x)}{x}\{1 + o(1)\}. \tag{295}$$

For every fixed $0 < \sigma < 1$, as $x \to \infty$,

$$\frac{\overline{\Phi}(\sigma x)}{\overline{\Phi}(x)} \sim \frac{1}{\sigma} \exp\left\{\frac{(1-\sigma^2)x^2}{2}\right\}. \tag{296}$$

For every $0 < H < \infty$, uniformly over $|u| \vee |v| \le H$, as $x \to \infty$,

$$\frac{p(x+u)}{p(x+v)} = \frac{x+v}{x+u} \exp\left\{-x(u-v) - \frac{u^2 - v^2}{2}\right\}\{1 + o(1)\}. \tag{297}$$

If $c > 0$ is fixed, then, as $q \downarrow 0$,

$$p^{-1}(cq) \sim \sqrt{2\log(1/q)}. \tag{298}$$

In particular, for every fixed $c_1, c_2 > 0$,

$$\frac{p^{-1}(c_1 q)}{p^{-1}(c_2 q)} \longrightarrow 1 \quad (q \downarrow 0).$$

Finally, for $x \ge 0$,

$$2\overline{\Phi}(x) \le e^{-x^2/2}. \tag{299}$$

For $x > 0$ and $\delta \ge 0$,

$$\frac{p(x+\delta)}{p(x)} \le \left(1 + x^{-2}\right) \exp\left\{-x\delta - \frac{\delta^2}{2}\right\}. \tag{300}$$

Consequently, if $x \to \infty$, then the following tail-ratio limits hold.

(i) *For every $0 < H < \infty$, uniformly over $|u| \vee |v| \le H$,*

$$\log \frac{p(x+u)}{p(x+v)} = -x(u-v) - \frac{u^2 - v^2}{2} + o(1). \tag{301}$$

(ii) *For every $0 < H < \infty$, uniformly over $|h| \le H$,*

$$\frac{p(x)}{p(x + h/x)} \longrightarrow e^h. \tag{302}$$

(iii) *If $0 < \delta_0 \le \delta \le \delta_1 < \infty$, then*

$$\frac{p(x+\delta)}{p(x)} \longrightarrow 0 \tag{303}$$

uniformly in δ.

(iv) *If $\delta_x \geq 0$ and $x\delta_x \to \infty$, then*

$$\frac{p(x+\delta_x)}{p(x)} \longrightarrow 0. \tag{304}$$

Proof. For $x > 0$, the upper Mills bound follows from

$$\overline{\Phi}(x) < \frac{1}{x}\int_x^\infty t\phi(t)\ \mathrm{d}t = \frac{\phi(x)}{x}. \tag{305}$$

Integration by parts gives

$$\overline{\Phi}(x) = \frac{\phi(x)}{x} - \int_x^\infty \frac{\phi(t)}{t^2}\ \mathrm{d}t > \frac{\phi(x)}{x} - \frac{\overline{\Phi}(x)}{x^2},$$

which proves the lower Mills bound

$$\frac{x}{1+x^2}\phi(x) < \overline{\Phi}(x). \tag{306}$$

A further integration by parts gives

$$\overline{\Phi}(x) = \frac{\phi(x)}{x} - \frac{\phi(x)}{x^3} + 3\int_x^\infty \frac{\phi(t)}{t^4}\,\mathrm{d}t,$$

where

$$0 < \int_x^\infty \frac{\phi(t)}{t^4}\,\mathrm{d}t < \frac{\phi(x)}{x^5}.$$

Hence

$$\overline{\Phi}(x) = \frac{\phi(x)}{x}\{1 - x^{-2} + O(x^{-4})\},$$

and inversion proves (294) in Lemma A.2. Equations (305) and (306) imply (293) in Lemma A.2 for $x > 0$. Since $\lambda(x) - x > 0$ also holds trivially for $x \leq 0$, the identity

$$\lambda'(x) = \lambda(x)\{\lambda(x) - x\}$$

shows that λ is increasing on $\mathbb{R}$.

Equations (305) and (306) give (295) in Lemma A.2. Applying this at x and σx proves (296) in Lemma A.2. Applying (295) in Lemma A.2 at $x+u$ and $x+v$ gives (297) in Lemma A.2, for every $0 < H < \infty$, uniformly over $|u| \vee |v| \leq H$. Moreover, as $x \to \infty$,

$$-\log p(x) = \frac{x^2}{2} + O(\log x),$$

which yields (298) in Lemma A.2 by inversion. To prove $2\overline{\Phi}(x) \leq e^{-x^2/2}$, set

$$d(x) = e^{-x^2/2} - 2\overline{\Phi}(x).$$

Then $d(0) = 0$, $\lim_{x\to\infty} d(x) = 0$, and

$$d'(x) = e^{-x^2/2}\left(\sqrt{\frac{2}{\pi}} - x\right).$$

Thus d first increases and then decreases to zero, so $d(x) \geq 0$ for all $x \geq 0$. Finally, (305) and (306) imply

$$\frac{p(x+\delta)}{p(x)} \leq \frac{1+x^2}{x(x+\delta)} \exp\left\{-x\delta - \frac{\delta^2}{2}\right\} \leq \left(1+x^{-2}\right) \exp\left\{-x\delta - \frac{\delta^2}{2}\right\},$$

which proves (300) in Lemma A.2.

It remains to prove the four stated tail-ratio consequences. Taking logarithms in (297) in Lemma A.2 gives (301) in Lemma A.2, because

$$\log \frac{x+v}{x+u} = o(1)$$

for every $0 < H < \infty$, uniformly over $|u| \vee |v| \leq H$. For (302) in Lemma A.2, apply (297) in Lemma A.2 with $u = 0$ and $v = h/x$. For every $0 < H < \infty$, uniformly over $|h| \leq H$,

$$\frac{p(x)}{p(x+h/x)} = \frac{x+h/x}{x} \exp\left\{h + \frac{h^2}{2x^2}\right\}\{1+o(1)\} \longrightarrow e^h.$$

For (303) in Lemma A.2, apply (301) in Lemma A.2 with $u = \delta$ and $v = 0$. If $0 < \delta_0 \leq \delta \leq \delta_1 < \infty$, then

$$\log \frac{p(x+\delta)}{p(x)} \leq -x\delta_0 + O(1) \longrightarrow -\infty$$

uniformly in δ. Finally, (304) in Lemma A.2 follows from (300) in Lemma A.2 with $\delta = \delta_x$, because $x\delta_x \to \infty$. □