No Universal Multiplicative FDR Bound for the Benjamini–Hochberg Procedure with Correlated Two-Sided Gaussian Tests

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Abstract

We study the worst-case false discovery rate of the Benjamini–Hochberg procedure applied to two-sided Gaussian p-values when the correlation matrix is otherwise unrestricted. Dobriban [2026] shows that BH does not always control the FDR at its nominal level. An analogous folklore conjecture is that BH controls the FDR up to a universal multiplicative constant. We prove that this conjecture is false. In particular, we construct Gaussian models for which the inflation factor $\mathrm{FDR}(\mathrm{BH}_q)/q$ diverges as $q \downarrow 0$. More precisely, for all sufficiently small q, the supremum over the number of hypotheses, mean vector, and correlation matrix is at least $cq\sqrt{\log(1/q)}$ for a universal constant $c > 0$. Finally, for a broad class of common-factor Gaussian models with arbitrary means and loadings, we prove the matching-order upper bound $\mathrm{FDR}(\mathrm{BH}_q) = O(q\sqrt{\log(1/q)})$, and hence the lower bound is sharp in order for this class.

1 Introduction

We observe a Gaussian vector $T = (T_1, \dots, T_m)$ with mean $\boldsymbol{\theta}$ and covariance matrix Σ, where $\Sigma_{ii} = 1$ for all i. The null hypotheses are $H_i : \theta_i = 0$, and their two-sided Gaussian p-values are $P_i = p(|T_i|)$, where

$$p(x) = 2\overline{\Phi}(x) = 2(1 - \Phi(x)),$$

and Φ is the cumulative distribution function (CDF) of the standard normal distribution $N(0, 1)$. We will also use $\phi = \Phi'$ to denote the density function throughout. The Benjamini–Hochberg procedure [Benjamini and Hochberg, 1995] at level q orders the p-values as

$$P_{(1)} \le \cdots \le P_{(m)}$$

and rejects the hypotheses corresponding to

$$P_{(1)}, \dots, P_{(R)}, \quad R = \max\{1 \le k \le m : P_{(k)} \le qk/m\},$$

with $R = 0$ if the set is empty. If V is the number of rejected true nulls, then

$$\mathrm{FDP} = \frac{V}{R \vee 1}, \quad \mathrm{FDR} = \mathbb{E}(\mathrm{FDP}).$$

There is a long-standing conjecture in the FDR literature that BH with nominal level q controls the FDR at level q regardless of the correlation matrix [Reiner-Benaim, 2007, Sarkar, 2023, Sarkar and Zhang, 2025]. These works stated or supported this conjecture. However, recently, Dobriban [2026] found a counterexample where $\mathrm{FDR} > 0.0104$ when $q = 0.01$.

In this paper, we ask a more refined question: how large can FDR be as a function of q? Formally, we seek a lower bound $\ell(q)$ such that

$$\sup_{\substack{m\in\mathbb{Z}_+,\ \boldsymbol{\theta}\in\mathbb{R}^m,\ \Sigma\in\mathbb{S}_+^m \\ \Sigma_{ii}=1,\ 1\le i\le m}} \mathrm{FDR}(\mathrm{BH}_q) \ge \ell(q). \tag{1}$$

Here $\mathbb{Z}_+ = \{1, 2, \dots\}$ and $\mathbb{S}_+^m$ denotes the cone of $m \times m$ symmetric positive semidefinite matrices. A relaxed folklore conjecture is that the worst-case FDR in (1) is upper bounded by Cq for some universal constant $C > 0$. However, we prove that one may take

$$\ell(q) = cq\sqrt{\log(1/q)}$$

for a universal constant $c > 0$. For every $c < 1/(2\sqrt{\pi})$, the bound holds for all sufficiently small q, where the required upper bound on q may depend on c. In particular, the FDR inflation factor FDR/q can diverge as $q \downarrow 0$.

This rate should be compared with the FDR-linking theorem of Su [2018]. Positive regression dependence within nulls (PRDN) relaxes positive regression dependence on a subset (PRDS) by imposing the relevant positive dependence condition only on the null p-values. Under PRDN, the FDR-linking argument gives an $O(q\log(1/q))$ bound. Section 3 proves a sharper bound for the common-factor Gaussian class considered here:

$$\mathrm{FDR}(\mathrm{BH}_q) = O(q\sqrt{\log(1/q)}),$$

uniformly over the number of hypotheses, means, and loadings. Thus, in this common-factor Gaussian class, the upper-bound rate improves on the general PRDN linking bound and matches the lower-bound order above. Moreover, as explained in Remark 2, the two-sided null p-values in this class satisfy the PRDN property by the absolute-value Gaussian MTP_2 criterion of Karlin and Rinott [1981]. Hence the upper bound sharpens the general PRDN-based linking rate within this common-factor subclass.

2 Lower bound

2.1 Construction

Fix $0 < q < 1$, fix $0 < \pi_0, \pi_1 < 1$ with $\pi_0 + \pi_1 = 1$, and fix $0 < a < 1$ and $\sigma = \sqrt{1-a^2}$. For each N, choose positive integers $m_{0,N}, m_{1,N}$ with $m_N = m_{0,N} + m_{1,N}$, $m_N \to \infty$, and

$$\frac{m_{0,N}}{m_N} \to \pi_0, \quad \frac{m_{1,N}}{m_N} \to \pi_1.$$

Given a probability measure ν_q on $\mathbb{R}$ for the Y-block means, draw

$$Z, \varepsilon_1, \varepsilon_2, \dots \overset{\text{iid}}{\sim} N(0,1), \quad \mu_1, \mu_2, \dots \overset{\text{iid}}{\sim} \nu_q,$$

with Z, the null-noise sequence, and the mean sequence mutually independent. Define the $m_{0,N}$ null and $m_{1,N}$ Y-block statistics by

$$X_i = aZ + \sigma\varepsilon_i \ (1 \le i \le m_{0,N}), \quad Y_j = \mu_j - Z \ (1 \le j \le m_{1,N}), \quad P_i = p(|X_i|), \quad P_{m_{0,N}+j} = p(|Y_j|).$$

Conditional on $\boldsymbol{\mu}_N := (\mu_1, \dots, \mu_{m_{1,N}})$, $(X_1, \dots, X_{m_{0,N}}, Y_1, \dots, Y_{m_{1,N}})$ is multivariate Gaussian with mean $(\mathbf{0}_{m_{0,N}}, \boldsymbol{\mu}_N)$.

Let $\mathrm{FDR}_{\boldsymbol{\mu}_N}(\mathrm{BH}_q)$ denote the actual FDR of the BH procedure applied to these statistics, conditional on $\boldsymbol{\mu}_N$. If, for some N, we can show

$$\mathbb{E}_{\boldsymbol{\mu}_N}\left[\mathrm{FDR}_{\boldsymbol{\mu}_N}(\mathrm{BH}_q)\right] \ge \ell(q), \tag{2}$$

then there exists a fixed vector $\boldsymbol{\mu}^*$ such that $\mathrm{FDR}_{\boldsymbol{\mu}^*}(\mathrm{BH}_q) \ge \ell(q)$.

2.2 Proof of the lower bound

We start by stating the normal-tail facts used below. We prove them for completeness.

Lemma 1. *The normal hazard $h(x) = \phi(x)/\overline{\Phi}(x)$ is increasing on $\mathbb{R}$ and satisfies*

$$x < h(x) < x + x^{-1} \quad (x > 0). \tag{3}$$

For every fixed $0 < \sigma < 1$, as $x \to \infty$,

$$\frac{\overline{\Phi}(\sigma x)}{\overline{\Phi}(x)} \sim \frac{1}{\sigma} \exp\left\{\frac{(1-\sigma^2)x^2}{2}\right\}. \tag{4}$$

If $c > 0$ is fixed, then, as $q \downarrow 0$,

$$p^{-1}(cq) \sim \sqrt{2\log(1/q)}. \tag{5}$$

In particular, for every fixed $c_1, c_2 > 0$,

$$\frac{p^{-1}(c_1 q)}{p^{-1}(c_2 q)} \longrightarrow 1 \qquad (q \downarrow 0).$$

Finally, for $x \geq 0$,

$$2\overline{\Phi}(x) \leq e^{-x^2/2}. \tag{6}$$

Proof. For $x > 0$, the upper Mills bound follows from

$$\overline{\Phi}(x) < \frac{1}{x}\int_x^\infty t\phi(t)\, dt = \frac{\phi(x)}{x}.$$

Integration by parts gives

$$\overline{\Phi}(x) = \frac{\phi(x)}{x} - \int_x^\infty \frac{\phi(t)}{t^2}\, dt > \frac{\phi(x)}{x} - \frac{\overline{\Phi}(x)}{x^2},$$

which proves the lower Mills bound

$$\frac{x}{1+x^2}\phi(x) < \overline{\Phi}(x).$$

The two Mills bounds imply (3) for $x > 0$. Since $h(x) - x > 0$ also holds trivially for $x \leq 0$, the identity $h'(x) = h(x)\{h(x) - x\}$ shows that h is increasing on $\mathbb{R}$.

The same Mills bounds give $\overline{\Phi}(x) \sim \phi(x)/x$. Applying this at x and σx proves (4). Moreover, as $x \to \infty$,

$$-\log p(x) = \frac{x^2}{2} + O(\log x),$$

which yields (5) by inversion. To prove $2\overline{\Phi}(x) \leq e^{-x^2/2}$, set

$$d(x) = e^{-x^2/2} - 2\overline{\Phi}(x).$$

Then $d(0) = 0$, $\lim_{x\to\infty} d(x) = 0$, and

$$d'(x) = e^{-x^2/2}\left(\sqrt{\frac{2}{\pi}} - x\right).$$

Thus d first increases and then decreases to zero, so $d(x) \geq 0$ for all $x \geq 0$. □

We next prove a lemma giving a liminf lower comparison between the finite-sample FDR and its population counterpart as $N \to \infty$. It is similar in spirit to the empirical-process approach to FDR control by Storey et al. [2004]. A similar argument is also used in Dobriban [2026].

Lemma 2. *Fix $0 < q < 1$ and a probability measure ν_q on $\mathbb{R}$. Fix $\epsilon > 0$. For $z \in \mathbb{R}$ and $0 \le s \le 1$, define*

$$R_{0,z}(s) = \mathbb{P}\{p(|X_i|) \le s \mid Z = z\}, \quad R_{1,z}(s) = \mathbb{P}\{p(|Y_U|) \le s \mid Z = z\}, \tag{7}$$

where $Y_U = U - Z$ and $U \sim \nu_q$ is independent of Z. Set, pointwise in s,

$$R_z(s) = \pi_0 R_{0,z}(s) + \pi_1 R_{1,z}(s). \tag{8}$$

Let $I_\epsilon \subset \mathbb{R}$ be a nonempty compact interval and let $t_z \in (0, q/(1+\epsilon))$ be a continuous function of $z \in I_\epsilon$, bounded away from zero. Suppose

$$\inf_{z \in I_\epsilon} \{qR_z(t_z) - t_z\} > 0 \tag{9}$$

and that, for some $\delta_\epsilon > 0$,

$$qR_z(s) \le (1 - \delta_\epsilon)s, \quad z \in I_\epsilon, \quad (1+\epsilon)t_z \le s \le q. \tag{10}$$

Then, for the finite models in the construction,

$$\liminf_{N\to\infty} \frac{\mathbb{E}_{\boldsymbol{\mu}_N}[\mathrm{FDR}_{\boldsymbol{\mu}_N}(\mathrm{BH}_q)]}{q} \ge \int_{I_\epsilon} \frac{\pi_0 R_{0,z}(t_z)}{(1+\epsilon)t_z} \phi(z)\, dz. \tag{11}$$

Proof. Conditional on $Z = z$, set

$$\widehat{R}_{0,N,z}(s) = \frac{1}{m_{0,N}} \sum_{i=1}^{m_{0,N}} \mathbf{1}\{p(|az + \sigma\varepsilon_i|) \le s\},$$
$$\widehat{R}_{1,N,z}(s) = \frac{1}{m_{1,N}} \sum_{j=1}^{m_{1,N}} \mathbf{1}\{p(|\mu_j - z|) \le s\}.$$

Set

$$\widehat{R}_{N,z} = \frac{m_{0,N}}{m_N} \widehat{R}_{0,N,z} + \frac{m_{1,N}}{m_N} \widehat{R}_{1,N,z}.$$

For $x \in \mathbb{R}$, define the empirical CDFs of the null noises and sampled Y-block means by

$$F^{\varepsilon}_{0,N}(x) = \frac{1}{m_{0,N}} \sum_{i=1}^{m_{0,N}} \mathbf{1}\{\varepsilon_i \le x\}, \quad F^{\mu}_{1,N}(x) = \frac{1}{m_{1,N}} \sum_{j=1}^{m_{1,N}} \mathbf{1}\{\mu_j \le x\}.$$

Let F_{ν_q} be the CDF of ν_q. Glivenko–Cantelli gives, almost surely,

$$\sup_x |F^{\varepsilon}_{0,N}(x) - \Phi(x)| \to 0, \quad \sup_x |F^{\mu}_{1,N}(x) - F_{\nu_q}(x)| \to 0.$$

For $0 < s \le q$, put $u_s = p^{-1}(s)$. Conditional on $Z = z$,

$$p(|az + \sigma\varepsilon|) \le s \quad \Longleftrightarrow \quad \varepsilon \le \alpha_{z,s} \ \text{ or } \ \varepsilon \ge \beta_{z,s},$$

where

$$\alpha_{z,s} = \frac{-u_s - az}{\sigma}, \quad \beta_{z,s} = \frac{u_s - az}{\sigma}.$$

Thus the set of ε-values satisfying the event is $(-\infty, \alpha_{z,s}] \cup [\beta_{z,s}, \infty)$. For any two CDFs $F, \widetilde{F}$ and any $\alpha \le \beta$,

$$\left| \{F(\alpha) + 1 - F(\beta-)\} - \{\widetilde{F}(\alpha) + 1 - \widetilde{F}(\beta-)\} \right| \le 2 \sup_x |F(x) - \widetilde{F}(x)|.$$

The case $s = 0$ is trivial. Applying this with $F = F^{\varepsilon}_{0,N}$ and $\widetilde{F} = \Phi$ gives

$$\sup_{z\in\mathbb{R},\, 0\le s\le q} |\widehat{R}_{0,N,z}(s) - R_{0,z}(s)| \le 2\sup_x |F^{\varepsilon}_{0,N}(x) - \Phi(x)| \to 0,$$

and applying the same argument to $\{p(|\mu - z|) \le s\} = \{\mu \le z - u_s\} \cup \{\mu \ge z + u_s\}$ gives

$$\sup_{z\in\mathbb{R},\, 0\le s\le q} |\widehat{R}_{1,N,z}(s) - R_{1,z}(s)| \le 2\sup_x |F^{\mu}_{1,N}(x) - F_{\nu_q}(x)| \to 0.$$

Thus $\widehat{R}_{N,z}$ converges uniformly to R_z over $z \in I_\epsilon$ and $0 \le s \le q$, almost surely. Indeed, with $w_N = m_{0,N}/m_N$,

$$\sup_{z,s} \left|\widehat{R}_{N,z}(s) - R_z(s)\right| \le 2w_N \sup_x |F^{\varepsilon}_{0,N}(x) - \Phi(x)| + 2(1 - w_N) \sup_x |F^{\mu}_{1,N}(x) - F_{\nu_q}(x)| \\ + |w_N - \pi_0|,$$

where the supremum on the left is over $z \in I_\epsilon$ and $0 \le s \le q$. Since $m_N \to \infty$ and $\pi_0, \pi_1 > 0$, both $m_{0,N}$ and $m_{1,N}$ diverge, so the right-hand side tends to zero almost surely.

Let

$$\eta_N = \sup_{z\in I_\epsilon,\, 0\le s\le q} |\widehat{R}_{N,z}(s) - R_z(s)|.$$

Almost surely, $\eta_N \to 0$. Let

$$\Delta_- = \inf_{z\in I_\epsilon} \{qR_z(t_z) - t_z\} > 0, \quad t_* = \inf_{z\in I_\epsilon} t_z > 0.$$

For all large N, $q\eta_N < \Delta_-/2$ and $q\eta_N < \delta_\epsilon t_*$. Then, uniformly in $z \in I_\epsilon$,

$$\widehat{R}_{N,z}(t_z) \ge R_z(t_z) - \eta_N > \frac{t_z}{q},$$

and, if $(1 + \epsilon)t_z \le s \le q$, then

$$\widehat{R}_{N,z}(s) \le R_z(s) + \eta_N \le (1 - \delta_\epsilon)\frac{s}{q} + \eta_N < (1 - \delta_\epsilon)\frac{s}{q} + \frac{\delta_\epsilon t_*}{q} \le (1 - \delta_\epsilon)\frac{s}{q} + \frac{\delta_\epsilon s}{(1 + \epsilon)q} = \left(1 - \frac{\epsilon\delta_\epsilon}{1 + \epsilon}\right)\frac{s}{q} < \frac{s}{q},$$

where the third inequality uses $q\eta_N < \delta_\epsilon t_*$ and the fourth uses $s \ge (1+\epsilon)t_z \ge (1+\epsilon)t_*$. Let $R_N(z)$ be the BH rejection count and put $\widehat{\tau}_N(z) = qR_N(z)/m_N$. With $k_{N,z} = m_N\widehat{R}_{N,z}(t_z)$, the inequality $\widehat{R}_{N,z}(t_z) > t_z/q$ gives $qk_{N,z}/m_N > t_z$ and makes the grid point $qk_{N,z}/m_N$ BH-admissible. Also, $\widehat{R}_{N,z}(s) < s/q$ for every $s \in [(1 + \epsilon)t_z, q]$, so no BH-admissible grid point lies in that interval. Hence, almost surely, for all sufficiently large N and all $z \in I_\epsilon$,

$$t_z \le \widehat{\tau}_N(z) < (1 + \epsilon)t_z.$$

By maximality in the step-up definition, $m_N\widehat{R}_{N,z}(\widehat{\tau}_N(z)) = R_N(z)$, even when some p-values are tied. Let FDP_N denote the finite-sample false discovery proportion. Hence, on $Z = z$,

$$\frac{\mathrm{FDP}_N}{q} \ge \frac{(m_{0,N}/m_N)\widehat{R}_{0,N,z}(\widehat{\tau}_N(z))}{\widehat{\tau}_N(z)} \ge \frac{(m_{0,N}/m_N)\widehat{R}_{0,N,z}(t_z)}{(1 + \epsilon)t_z}.$$

The first inequality keeps only false discoveries from the X-block; if ν_q assigns mass to zero, nulls in the Y-block can only increase the FDP. The averaged conditional FDR is the full expectation of this FDP:

$$\mathbb{E}_{\boldsymbol{\mu}_N}[\mathrm{FDR}_{\boldsymbol{\mu}_N}(\mathrm{BH}_q)] = \mathbb{E}_{\boldsymbol{\mu}_N,\varepsilon,Z}[\mathrm{FDP}_N].$$

Define

$$G(z) = \begin{cases} \dfrac{\pi_0 R_{0,z}(t_z)}{(1+\epsilon)t_z}, & z \in I_\epsilon, \\ 0, & z \notin I_\epsilon. \end{cases}$$

Since the lower bound above holds for every $z \in I_\epsilon$ and the FDP is nonnegative, the almost sure uniform convergence gives

$$\liminf_{N\to\infty} \frac{\mathrm{FDP}_N}{q} \geq G(Z).$$

Fatou's lemma therefore yields

$$\begin{aligned} \liminf_{N\to\infty} \frac{\mathbb{E}_{\boldsymbol{\mu}_N}[\mathrm{FDR}_{\boldsymbol{\mu}_N}(\mathrm{BH}_q)]}{q} &= \liminf_{N\to\infty} \mathbb{E}_{\boldsymbol{\mu}_N,\varepsilon,Z}\left[\frac{\mathrm{FDP}_N}{q}\right] \\ &\geq \mathbb{E}_Z\left[G(Z)\right] \\ &= \int_{I_\epsilon} \frac{\pi_0 R_{0,z}(t_z)}{(1+\epsilon)t_z}\phi(z)\,dz, \end{aligned}$$

which is the claim. □

The following theorem gives a closed-form lower bound for the worst-case FDR at small nominal levels.

Theorem 1. *For every $\delta > 0$, there exists $q_0 = q_0(\delta) \in (0,1)$ such that, for all $0 < q < q_0$,*

$$\sup_{\substack{m\in\mathbb{Z}_+,\ \boldsymbol{\theta}\in\mathbb{R}^m,\ \Sigma\in\mathbb{S}_+^m \\ \Sigma_{ii}=1,\ 1\leq i\leq m}} \mathrm{FDR}(\mathrm{BH}_q) \geq \left(\frac{1}{2\sqrt{\pi}} - \delta\right) q\sqrt{\log(1/q)}.$$

Proof. Fix $\delta > 0$. Choose fixed parameters

$$0 < \pi_1 < 1, \quad 0 < a < b < c < 1, \quad \epsilon > 0$$

so that, with $\pi_0 = 1 - \pi_1$ and $\sigma = \sqrt{1-a^2}$,

$$\frac{\pi_0(c-b)}{2(1+\epsilon)\sigma\sqrt{\pi}} > \frac{1}{2\sqrt{\pi}} - \frac{\delta}{2}. \tag{12}$$

This is possible by first taking π_1 small, then a small, then $b > a$ sufficiently close to a, then $c < 1$ sufficiently close to one, and finally ϵ small. Put $\kappa = a/(1+a)$, choose $0 < \alpha < a/c$, define

$$\rho_* := (1+\epsilon)^{-\kappa\alpha} < 1, \tag{13}$$

and choose $\eta_* > 0$ such that $\rho_* + \eta_* < 1$. We choose $q_0 \in (0,1)$ below by imposing finitely many small-q conditions. All asymptotics in the proof are as $q \downarrow 0$ with these fixed parameters. Let

$$u_q = p^{-1}(\pi_1 q) = \Phi^{-1}\left(1 - \frac{\pi_1 q}{2}\right), \quad \tilde{u}_q = p^{-1}(q) = \Phi^{-1}\left(1 - \frac{q}{2}\right),$$

By (5),

$$\frac{\tilde{u}_q}{u_q} \to 1. \tag{14}$$

Since $a < b$ and $c < 1 < 1+a$, (14) implies that, after decreasing q_0 if necessary, for all $0 < q < q_0$,

$$1 + a - \frac{\tilde{u}_q}{u_q} < b < c < 1 + a. \tag{15}$$

Moreover, because $r := b/a > 1$, applying (4) with argument ru_q and scale $1/r$ gives

$$\frac{p(ru_q)}{p(u_q)} \sim \frac{1}{r}\exp\left\{-\frac{(r^2-1)u_q^2}{2}\right\} \to 0.$$

Therefore

$$\frac{(1+\epsilon)p((b/a)u_q)}{q} = (1+\epsilon)\pi_1\frac{p((b/a)u_q)}{p(u_q)} \to 0.$$

Thus, after decreasing q_0 if necessary, for all $0 < q < q_0$,

$$(1+\epsilon)p((b/a)u_q) < q. \tag{16}$$

Again by (5),

$$\frac{\tilde{u}_q}{(c/a)u_q + ((b/a)u_q)^{-1}} \to \frac{a}{c}. \tag{17}$$

By the already fixed choice $0 < \alpha < a/c$, decrease q_0 if necessary so that, for all $0 < q < q_0$,

$$\frac{\tilde{u}_q}{(c/a)u_q + ((b/a)u_q)^{-1}} \geq \alpha. \tag{18}$$

Conditional on $Z = z$, the density f_z of $az + \sigma\varepsilon$ satisfies

$$\frac{f_z(x)}{\phi(x)} = \frac{1}{\sigma}\exp\left\{\frac{z^2}{2} - \frac{(ax-z)^2}{2\sigma^2}\right\} \leq \frac{1}{\sigma}e^{z^2/2}. \tag{19}$$

Thus, uniformly over $0 \leq z \leq c\,u_q$ and $v \geq \tilde{u}_q$,

$$\frac{q\pi_0}{p(v)}\mathbb{P}\{|az+\sigma\varepsilon| \geq v \mid Z = z\} \leq \frac{q\pi_0}{\sigma}e^{c^2u_q^2/2} \leq \frac{\pi_0}{\sigma\pi_1}e^{-(1-c^2)u_q^2/2} \to 0, \tag{20}$$

where the last inequality uses $p(u_q) = \pi_1 q \leq e^{-u_q^2/2}$. Since the deterministic upper bound in (20) tends to zero, decrease q_0 if necessary so that, for all $0 < q < q_0$,

$$\frac{q\pi_0}{\sigma}e^{c^2u_q^2/2} \leq \eta_*. \tag{21}$$

Fix such a q. Let

$$I_\epsilon = [b\,u_q, c\,u_q], \quad t_z = p(z/a), \quad \delta_\epsilon = 1 - \rho_* - \eta_* > 0.$$

The positivity of δ_ϵ follows from (13) and the choice of η_*. There is a probability measure ν_q, supported on $[(1+a)u_q, \infty)$, and uniquely determined by

$$\nu_q([(1+a)x, \infty)) = \frac{p(x)}{\pi_1 q}, \quad x \geq u_q. \tag{22}$$

Indeed,

$$\frac{p(u_q)}{\pi_1 q} = 1, \quad \frac{p(x)}{\pi_1 q} \downarrow 0 \quad \text{as } x \to \infty,$$

and $x \mapsto p(x)/(\pi_1 q)$ is continuous and strictly decreasing.

Use the notation $R_{0,z}, R_{1,z}, R_z$ from (7)–(8) of Lemma 2 for this choice of ν_q. We verify (9) and (10). Let $z \in I_\epsilon$. By (15),

$$z + v \geq b\,u_q + \tilde{u}_q > (1+a)u_q \quad \text{whenever } p(v) \leq q. \tag{23}$$

Also, if $\mu \in \operatorname{supp}(\nu_q)$, then

$$Y_\mu = \mu - z \geq (1+a)u_q - c\,u_q = (1+a-c)u_q > 0.$$

The last inequality uses (15). For $p(v) \le q$, put $w = (z+v)/(1+a)$. By (23), $w \ge u_q$. Conditional on $Z = z$, $Y_U = U - z$. Since

$$U - z \ge (1+a)u_q - c\,u_q = (1+a-c)u_q > 0,$$

we have $|Y_U| = U - z$. Since p is decreasing,

$$\{p(|Y_U|) \le p(v)\} = \{U \ge z + v\} = \{U \ge (1+a)w\}.$$

Therefore, by (22),

$$R_{1,z}(p(v)) = \nu_q([(1+a)w, \infty)) = \frac{p(w)}{\pi_1 q}, \quad w = \frac{z+v}{1+a}. \tag{24}$$

We may substitute $v = z/a$ in (24). Indeed,

$$\frac{z}{a} \ge \frac{b}{a} u_q > u_q > \tilde{u}_q$$

for $z \in I_\epsilon$, so $p(z/a) < q$. Here $(b/a)u_q > u_q$ follows from $b > a$, while $u_q > \tilde{u}_q$ follows from $\pi_1 < 1$ and the monotonicity of p^{-1}. Taking $v = z/a$ in (24) gives $w = z/a$. Since $t_z = p(z/a)$,

$$\pi_1 R_{1,z}(t_z) = \frac{t_z}{q},$$

and thus

$$qR_z(t_z) - t_z = q\pi_0 R_{0,z}(t_z) > 0. \tag{25}$$

Next let $p(v) \in [(1+\epsilon)t_z, q]$. Since p is decreasing and $t_z = p(z/a)$,

$$p(v) \ge (1+\epsilon)t_z > t_z = p(z/a) \quad \Longrightarrow \quad v \le z/a.$$

For this v, the value of w in (24) is

$$w = \frac{z+v}{1+a} = v + \frac{a}{1+a}\left(\frac{z}{a} - v\right) = v + \kappa\left(\frac{z}{a} - v\right),$$

so $v \le w \le z/a$. By (3) of Lemma 1, the normal hazard $h(x) = \phi(x)/\overline{\Phi}(x)$ is increasing and $x < h(x) < x + x^{-1}$ for $x > 0$. Since $v \ge \tilde{u}_q$ and $z/a \in [(b/a)u_q, (c/a)u_q]$,

$$\frac{h(v)}{h(z/a)} \ge \frac{v}{z/a + (z/a)^{-1}} \ge \frac{\tilde{u}_q}{(c/a)u_q + ((b/a)u_q)^{-1}} \ge \alpha, \tag{26}$$

where the last inequality is (18). Also, $h(x) \le h(z/a)$ for $v \le x \le z/a$, because h is increasing. Hence $\int_v^{z/a} h(x)\,dx \le (z/a - v)h(z/a)$, while (26) gives $h(v) \ge \alpha h(z/a)$. Consequently,

$$\begin{aligned}
\log\frac{p(v)}{p(w)} &= \int_v^w h(x)\,dx \ge (w-v)h(v) = \kappa(z/a - v)h(v) \\
&\ge \kappa\alpha \int_v^{z/a} h(x)\,dx = \kappa\alpha \log\frac{p(v)}{p(z/a)} \ge \kappa\alpha\log(1+\epsilon).
\end{aligned}$$

Using (24) and the bound above, and then taking the supremum over the arbitrary choices of $z \in I_\epsilon$ and $(1+\epsilon)t_z \le s = p(v) \le q$, gives

$$\sup_{z \in I_\epsilon} \sup_{(1+\epsilon)t_z \le s \le q} \frac{q\pi_1 R_{1,z}(s)}{s} \le (1+\epsilon)^{-\kappa\alpha} = \rho_*. \tag{27}$$

where the last equality uses (13).

For $0 < s \le q$, write $s = p(v)$, so $v \ge \tilde{u}_q$. Conditional on $Z = z$,

$$\{p(|az + \sigma\varepsilon|) \le p(v)\} = \{|az + \sigma\varepsilon| \ge v\}.$$

Thus

$$R_{0,z}(p(v)) = \overline{\Phi}\left(\frac{v - az}{\sigma}\right) + \overline{\Phi}\left(\frac{v + az}{\sigma}\right). \tag{28}$$

For $z \in I_\epsilon$, we have $0 \le z \le cu_q$. Therefore (19) gives, for $s = p(v) \le q$,

$$R_{0,z}(s) = \int_{|x| \ge v} f_z(x)\, dx \le \frac{1}{\sigma} e^{z^2/2} \int_{|x| \ge v} \phi(x)\, dx = \frac{1}{\sigma} e^{z^2/2} s.$$

Hence, by (21),

$$\frac{q\pi_0 R_{0,z}(s)}{s} \le \frac{q\pi_0}{\sigma} e^{z^2/2} \le \frac{q\pi_0}{\sigma} e^{c^2 u_q^2/2} \le \eta_*.$$

Taking the supremum over $z \in I_\epsilon$ and $0 < s \le q$ yields

$$\sup_{z \in I_\epsilon} \sup_{0 < s \le q} \frac{q\pi_0 R_{0,z}(s)}{s} \le \eta_*. \tag{29}$$

Combining (27) and (29) gives

$$qR_z(s) \le (1 - \delta_\epsilon)s, \quad z \in I_\epsilon, \quad (1 + \epsilon)t_z \le s \le q. \tag{30}$$

Also, since p is decreasing and $\sigma^2 = 1 - a^2$,

$$\begin{aligned} R_{0,z}(p(z/a)) &= \mathbb{P}\{|az + \sigma\varepsilon| \ge z/a\} \\ &= \mathbb{P}\left(\varepsilon \ge \frac{(1 - a^2)z}{a\sigma}\right) + \mathbb{P}\left(\varepsilon \le -\frac{(1 + a^2)z}{a\sigma}\right) \\ &= \overline{\Phi}(\sigma z/a) + \overline{\Phi}\left(\frac{1 + a^2}{a\sigma} z\right). \end{aligned} \tag{31}$$

The function t_z is continuous and, for fixed q, bounded away from zero on I_ϵ. By (16), $(1 + \epsilon)t_z \le (1 + \epsilon)p((b/a)u_q) < q$. The lower margin (9) follows from (25), whose minimum over I_ϵ is positive, and the upper margin (10) follows from (30). Therefore the conclusion (11) of Lemma 2 gives

$$\liminf_{N \to \infty} \frac{\mathbb{E}_{\boldsymbol{\mu}_N}[\mathrm{FDR}_{\boldsymbol{\mu}_N}(\mathrm{BH}_q)]}{q} \ge \int_{b\, u_q}^{c\, u_q} \frac{\pi_0 R_{0,z}(p(z/a))}{(1 + \epsilon)p(z/a)} \phi(z)\, dz. \tag{32}$$

Plugging (31) into (32) and using only the first summand gives

$$\liminf_{N \to \infty} \frac{\mathbb{E}_{\boldsymbol{\mu}_N}[\mathrm{FDR}_{\boldsymbol{\mu}_N}(\mathrm{BH}_q)]}{q} \ge \frac{\pi_0}{2(1 + \epsilon)} \int_{b\, u_q}^{c\, u_q} \frac{\overline{\Phi}(\sigma z/a)}{\overline{\Phi}(z/a)} \phi(z)\, dz. \tag{33}$$

The Mills-ratio asymptotic (4) gives, uniformly for $z \in [b\, u_q, c\, u_q]$,

$$\frac{\overline{\Phi}(\sigma z/a)}{\overline{\Phi}(z/a)} = (1 + o(1)) \frac{1}{\sigma} e^{z^2/2}.$$

The $o(1)$ term is as $q \downarrow 0$, with the fixed parameters chosen above. The uniformity follows from

$$\overline{\Phi}(x) = \frac{\phi(x)}{x}\{1 + O(x^{-2})\}, \quad x \ge M.$$

Since $z/a \geq b\,u_q/a \to \infty$, both arguments in the ratio are uniformly large on the integration interval. As $\phi(z) = (2\pi)^{-1/2} e^{-z^2/2}$,

$$\liminf_{N\to\infty} \frac{\mathbb{E}_{\boldsymbol{\mu}_N}[\mathrm{FDR}_{\boldsymbol{\mu}_N}(\mathrm{BH}_q)]}{q\,u_q} \geq \{1+o(1)\} \frac{\pi_0(c-b)}{2(1+\epsilon)\sigma\sqrt{2\pi}}.$$

The $o(1)$ in the last display is again as $q \downarrow 0$. Since $u_q \sim \sqrt{2\log(1/q)}$ by (5), the coefficient of $q\sqrt{\log(1/q)}$ is asymptotically

$$\frac{\pi_0(c-b)}{2(1+\epsilon)\sigma\sqrt{\pi}}.$$

By (12), after decreasing q_0 one final time so that the remaining $o(1)$ loss is smaller than the unused $\delta/2$ slack, for all $0 < q < q_0$,

$$\liminf_{N\to\infty} \mathbb{E}_{\boldsymbol{\mu}_N}[\mathrm{FDR}_{\boldsymbol{\mu}_N}(\mathrm{BH}_q)] \geq \left(\frac{1}{2\sqrt{\pi}} - \delta\right) q\sqrt{\log(1/q)}.$$

Therefore, for every $\xi > 0$, some finite N satisfies

$$\mathbb{E}_{\boldsymbol{\mu}_N}[\mathrm{FDR}_{\boldsymbol{\mu}_N}(\mathrm{BH}_q)] \geq \left(\frac{1}{2\sqrt{\pi}} - \delta\right) q\sqrt{\log(1/q)} - \xi.$$

For that N, there exists a deterministic realization $\boldsymbol{\mu}^*$ with

$$\mathrm{FDR}_{\boldsymbol{\mu}^*}(\mathrm{BH}_q) \geq \left(\frac{1}{2\sqrt{\pi}} - \delta\right) q\sqrt{\log(1/q)} - \xi.$$

Because the expectation over $\boldsymbol{\mu}_N$ is an average over deterministic mean vectors, this deterministic realization exists. Letting $\xi \downarrow 0$ gives the asserted supremum bound. □

Remark 1 (Approaching the lower bound with a positive-definite covariance matrix). Fix N and a realized mean vector $\boldsymbol{\mu}^* = (\mu_1^*, \ldots, \mu_{m_{1,N}}^*)$. Keep the nulls $X_i = aZ + \sigma\varepsilon_i$ unchanged, and replace $Y_j = \mu_j^* - Z$ by

$$Y_j^{(\delta)} = \mu_j^* - \sqrt{1-\delta^2}\, Z + \delta\eta_j, \quad 1 \leq j \leq m_{1,N},$$

where the η_j's are iid standard normals, independent of $Z, \varepsilon_1, \ldots, \varepsilon_{m_{0,N}}$. The covariance matrix has the form

$$\Sigma_\delta = \lambda_\delta \lambda_\delta^\top + D_\delta, \quad \lambda_\delta = \begin{pmatrix} a\mathbf{1}_{m_{0,N}} \\ -\sqrt{1-\delta^2}\mathbf{1}_{m_{1,N}} \end{pmatrix},$$

with

$$D_\delta = \mathrm{diag}(\sigma^2 \mathbf{1}_{m_{0,N}}, \delta^2 \mathbf{1}_{m_{1,N}}).$$

Thus, for every $0 < \delta \leq 1$,

$$\Sigma_\delta \succeq D_\delta \succ 0.$$

Let $X = (X_1, \ldots, X_{m_{0,N}})$, $Y^{(\delta)} = (Y_1^{(\delta)}, \ldots, Y_{m_{1,N}}^{(\delta)})$, and let $P^{(\delta)}$ be the vector of two-sided p-values computed from $(X, Y^{(\delta)})$. Let $\mathrm{FDP}^{(\delta)}$ and $\mathrm{FDR}_{\boldsymbol{\mu}^*}^{(\delta)}(\mathrm{BH}_q)$ denote the FDP and FDR of BH for this perturbed model. Under the natural coupling, as $\delta \downarrow 0$,

$$(X, Y^{(\delta)}) \to (X, Y^{(0)}) \quad \text{a.s.,} \quad P^{(\delta)} \to P^{(0)} \quad \text{a.s.}$$

Let

$$\mathcal{E} = \left\{\exists\, 1 \leq i, k \leq m_N : P_i^{(0)} = \frac{qk}{m_N}\right\}.$$

Each $P_i^{(0)}$ has a continuous law, hence $\mathbb{P}(\mathcal{E}) = 0$. On $\mathcal{E}^c$, the finite set of BH comparisons is locally constant at $P^{(0)}$. Therefore

$$\mathrm{FDP}^{(\delta)} \to \mathrm{FDP}^{(0)} \quad \text{a.s. as } \delta \downarrow 0, \quad 0 \le \mathrm{FDP}^{(\delta)} \le 1.$$

Dominated convergence gives

$$\mathrm{FDR}^{(\delta)}_{\boldsymbol{\mu}^*}(\mathrm{BH}_q) \to \mathrm{FDR}^{(0)}_{\boldsymbol{\mu}^*}(\mathrm{BH}_q) \quad \text{as } \delta \downarrow 0.$$

Thus, any strict lower bound obtained in the singular construction can be approached by restricting Σ to be positive definite.

3 Upper bound for a class of common-factor models

In this section, we examine a class of common-factor models that contains the construction in Subsection 2.1, with null loading a and Y-block loading -1. We show that the upper bound is $O(q\sqrt{\log(1/q)})$, which gives matching orders for small q.

Theorem 2. *Let*

$$T_i = \mu_i + a_i Z + \sqrt{1 - a_i^2}\,\varepsilon_i, \quad 1 \le i \le m,$$

where $Z, \varepsilon_1, \ldots, \varepsilon_m$ are independent standard normal variables and $|a_i| \le 1$. Form the two-sided Gaussian p-values $P_i = p(|T_i|)$ and apply BH at level $0 < q \le 2\overline{\Phi}(1) \approx 0.3173$. Let $\mathcal{H}_0 = \{i : \mu_i = 0\}$, $m_0 = |\mathcal{H}_0|$, and $\pi_0 = m_0/m$. If u_q is defined by $2\overline{\Phi}(u_q) = q$, then $\mathrm{FDR}(\mathrm{BH}_q) = 0$ when $m_0 = 0$, and otherwise

$$\mathrm{FDR}(\mathrm{BH}_q) \le q\left\{1 + \pi_0\left(\frac{1-q}{2} + \frac{u_q}{\sqrt{2\pi}}\right)\right\}.$$

Consequently the worst FDR in this common-factor class is $O\{q\sqrt{\log(1/q)}\}$, uniformly in m, the means $(\mu_1, \ldots, \mu_m)$, and the loadings $(a_1, \ldots, a_m)$, as $q \downarrow 0$.

Remark 2 (PRDN for the two-sided null p-values)*.* The theorem imposes no sign restriction on the loadings a_i, so the null correlations $a_i a_j$ may be negative. Nevertheless, the two-sided null p-values satisfy PRDN. Let $T_0 = (T_i : i \in \mathcal{H}_0)$ be the null subvector, let $a_0 = (a_i : i \in \mathcal{H}_0)$, and put

$$D_0 = \mathrm{diag}(1 - a_i^2 : i \in \mathcal{H}_0).$$

The null covariance matrix is

$$\Sigma_0 = D_0 + a_0 a_0^\top.$$

Assume first that $|a_i| < 1$ for every null coordinate. Let

$$B = \mathrm{diag}\{\mathrm{sign}(a_i) : i \in \mathcal{H}_0\}, \quad r = |a_0|,$$

with $\mathrm{sign}(0) = 1$. Then

$$B\Sigma_0 B = D_0 + rr^\top.$$

By the Sherman–Morrison formula,

$$(D_0 + rr^\top)^{-1} = D_0^{-1} - \frac{D_0^{-1} rr^\top D_0^{-1}}{1 + r^\top D_0^{-1} r}.$$

Thus every off-diagonal entry of $(D_0 + rr^\top)^{-1}$ is nonpositive. The Karlin–Rinott absolute-value multinormal MTP_2 criterion [Karlin and Rinott, 1981] therefore applies to BT_0, and $(|(BT_0)_i| : i \in \mathcal{H}_0)$ is MTP_2. Since $|(BT_0)_i| = |T_i|$, coordinatewise sign changes do not alter the two-sided p-values $p(|T_i|)$.

If some $|a_i| = 1$, define $a_i^{(\eta)} = (1-\eta)a_i$ for all $i \in \mathcal{H}_0$, and let $\eta \downarrow 0$. The corresponding absolute null vectors converge weakly to $(|T_i| : i \in \mathcal{H}_0)$. Under the measure-theoretic definition of MTP_2, the class of MTP_2 probability measures is closed under weak convergence [Colangelo et al., 2006]. Hence the limiting absolute-value law remains MTP_2. Appendix A.1 of Su [2018] then implies that the induced two-sided null p-values satisfy PRDN.

Proof. A leave-one-out form of BH self-consistency is standard in FDR proofs [Benjamini and Yekutieli, 2001]; we include the argument for completeness. If $m_0 = 0$, then $V = 0$ identically and hence $\text{FDR} = 0$. Assume henceforth that $m_0 \geq 1$. Fix a true null i. Let R_i be the BH rejection count after replacing its p-value by zero; in particular, $R_i \geq 1$. Pointwise in the p-value vector,

$$\frac{\mathbf{1}\{i \text{ is rejected}\}}{R \vee 1} = \frac{\mathbf{1}\{P_i \leq qR_i/m\}}{R_i}.$$

Indeed, if i is rejected, replacing P_i by zero does not change the step-up rejection count. Conversely, if $P_i \leq qR_i/m$, restoring P_i leaves at least R_i values at or below qR_i/m. Thus at least R_i values remain below the BH threshold qR_i/m, so BH makes at least R_i rejections. Coordinatewise monotonicity gives at most R_i rejections. Here ties cause no ambiguity: the BH rejection set is $\{j : P_j \leq qR/m\}$ and has exactly R elements. If more than R values were at most qR/m, then $P_{(R+1)} \leq qR/m < q(R+1)/m$, contradicting maximality of R. Summing gives the leave-one-out identity

$$\text{FDR} = \sum_{i \in \mathcal{H}_0} \mathbb{E}\left[\frac{\mathbf{1}\{P_i \leq qR_i/m\}}{R_i}\right]. \tag{34}$$

Conditional on $Z = z$, P_i is independent of R_i. Indeed, set $s_j = \sqrt{1-a_j^2}$ and let $\mathcal{R}(\cdot)$ denote the BH step-up rejection count. Then

$$R_i = \mathcal{R}(P_1, \ldots, P_{i-1}, 0, P_{i+1}, \ldots, P_m), \quad P_j = p(|\mu_j + a_j z + s_j \varepsilon_j|), \quad j \neq i.$$

Hence, conditional on $Z = z$, R_i is measurable with respect to $\sigma(\varepsilon_j : j \neq i,\ s_j > 0)$; coordinates with $s_j = 0$ are constants after conditioning. Meanwhile $P_i = p(|a_i z + s_i \varepsilon_i|)$ for a true null i, or is deterministic if $s_i = 0$. Since ε_i is independent of $\{\varepsilon_j : j \neq i\}$, the claim follows. With

$$F_{i,z}(t) = \mathbb{P}(P_i \leq t \mid Z = z),$$

and $0 < qR_i/m \leq q$,

$$\begin{aligned}
\mathbb{E}\left[\frac{\mathbf{1}\{P_i \leq qR_i/m\}}{R_i} \,\middle|\, Z = z\right] &= \mathbb{E}\left[\frac{1}{R_i}\mathbb{P}\left(P_i \leq \frac{qR_i}{m} \mid Z = z, R_i\right) \,\middle|\, Z = z\right] \\
&= \mathbb{E}\left[\frac{F_{i,z}(qR_i/m)}{R_i} \,\middle|\, Z = z\right] \\
&= \frac{q}{m}\,\mathbb{E}\left[\frac{F_{i,z}(qR_i/m)}{qR_i/m} \,\middle|\, Z = z\right] \\
&\leq \frac{q}{m} \sup_{0<t\leq q} \frac{F_{i,z}(t)}{t}.
\end{aligned}$$

Summing the preceding conditional bound over the true nulls gives

$$\mathbb{E}(\text{FDP} \mid Z = z) \leq \frac{q}{m} \sum_{i \in \mathcal{H}_0} \sup_{0<t\leq q} \frac{F_{i,z}(t)}{t}. \tag{35}$$

To average (35) over Z, it remains to control

$$\int_{-u_q}^{u_q} \phi(z) \sup_{0<t\le q} \frac{F_{i,z}(t)}{t}\, dz$$

for each true null i. We do this on $|z| \le u_q$; the event $|Z| > u_q$ is handled at the end by FDP ≤ 1. Fix a true-null index i. Put $r = |a_i|$ and $s = \sqrt{1-r^2}$, and first suppose $s > 0$. Conditional on $Z = z$, $T_i \sim N(a_i z, s^2)$. At the threshold $t = p(u) = 2\overline{\Phi}(u) \le q$, equivalently $u \ge u_q$, hence

$$F_{i,z}(p(u)) = \int_u^\infty s^{-1}\phi\{(x - a_i z)/s\}\, dx + \int_u^\infty s^{-1}\phi\{(x + a_i z)/s\}\, dx.$$

Since $s^2 = 1 - a_i^2$,

$$\begin{aligned} \frac{s^{-1}\phi\{(x-a_iz)/s\}}{\phi(x)} &= \frac{1}{s}\exp\left\{-\frac{(x-a_iz)^2}{2s^2} + \frac{x^2}{2}\right\} \\ &= \frac{1}{s}\exp\left\{\frac{z^2}{2} - \frac{(a_ix - z)^2}{2s^2}\right\}. \end{aligned}$$

The second tail is the same calculation with z replaced by $-z$. Choose $\eta_+, \eta_- \in \{-1, 1\}$ with $\{\eta_+, \eta_-\} = \{-1, 1\}$ so that $\eta_+ a_i z \ge 0$ and $\eta_- a_i z \le 0$. For $x \ge u$,

$$|a_i x - \eta_+ z| \ge (ru - |z|)_+, \qquad |a_i x - \eta_- z| \ge ru + |z|.$$

Both right-hand sides are nondecreasing in u, so for $u \ge u_q$,

$$\begin{aligned} \phi(z)\frac{F_{i,z}(p(u))}{p(u)} &\le \frac{\phi(z)}{p(u)}\frac{e^{z^2/2}}{s}\left[e^{-(ru_q-|z|)_+^2/(2s^2)} + e^{-(ru_q+|z|)^2/(2s^2)}\right]\int_u^\infty \phi(x)\, dx \\ &= \frac{1}{2s\sqrt{2\pi}}\left[e^{-(ru_q-|z|)_+^2/(2s^2)} + e^{-(ru_q+|z|)^2/(2s^2)}\right], \end{aligned}$$

because $\int_u^\infty \phi(x)\, dx = p(u)/2$. Taking the supremum gives

$$\phi(z)\sup_{0<t\le q}\frac{F_{i,z}(t)}{t} \le \frac{1}{2s\sqrt{2\pi}}\left[e^{-(ru_q-|z|)_+^2/(2s^2)} + e^{-(ru_q+|z|)^2/(2s^2)}\right]. \tag{36}$$

Integrating (36) over $|z| \le u_q$ and using symmetry gives

$$\int_{-u_q}^{u_q} \phi(z)\sup_{0<t\le q}\frac{F_{i,z}(t)}{t}\, dz \le \frac{1}{s\sqrt{2\pi}}\left\{\int_0^{u_q} e^{-(ru_q-z)_+^2/(2s^2)}\, dz + \int_0^{u_q} e^{-(ru_q+z)^2/(2s^2)}\, dz\right\}.$$

The two terms are

$$\begin{aligned} \int_0^{u_q} e^{-(ru_q-z)_+^2/(2s^2)}\, dz &= \int_0^{ru_q} e^{-y^2/(2s^2)}\, dy + (1-r)u_q, \\ \int_0^{u_q} e^{-(ru_q+z)^2/(2s^2)}\, dz &= \int_{ru_q}^{(1+r)u_q} e^{-y^2/(2s^2)}\, dy. \end{aligned}$$

Consequently, writing $\lambda = \sqrt{(1-r)/(1+r)} \in (0, 1]$,

$$\begin{aligned} \int_{-u_q}^{u_q} \phi(z)\sup_{0<t\le q}\frac{F_{i,z}(t)}{t}\, dz &\le \frac{1}{s\sqrt{2\pi}}\left\{\int_0^{(1+r)u_q} e^{-y^2/(2s^2)}\, dy + (1-r)u_q\right\} \\ &= \left\{\Phi\left(\frac{(1+r)u_q}{s}\right) - \frac{1}{2}\right\} + \frac{(1-r)u_q}{s\sqrt{2\pi}} \\ &= \Phi(u_q/\lambda) - \frac{1}{2} + \frac{u_q\lambda}{\sqrt{2\pi}}, \end{aligned} \tag{37}$$

where the last line uses $s^2 = 1 - r^2 = (1 - r)(1 + r)$, so $(1 + r)/s = 1/\lambda$ and $(1 - r)/s = \lambda$. The last expression is increasing in $\lambda \in (0, 1]$. Indeed, its derivative is

$$\frac{u_q}{\sqrt{2\pi}} \left[1 - \lambda^{-2} \exp\{-u_q^2/(2\lambda^2)\}\right] \geq 0,$$

because $u_q \geq 1$ and, on putting $w = \lambda^{-2} \geq 1$,

$$we^{-u_q^2 w/2} \leq we^{-w/2} \leq 2/e < 1.$$

Hence (37) is at most

$$\Phi(u_q) - \frac{1}{2} + \frac{u_q}{\sqrt{2\pi}} = \frac{1 - q}{2} + \frac{u_q}{\sqrt{2\pi}}. \tag{38}$$

When $r = 1$, the conditional p-value is deterministic; for $|z| < u_q$ it exceeds q, while the boundary $|z| = u_q$ has Lebesgue measure zero, so (38) remains valid. When $r = 0$, the conditional null p-value is uniform and independent of Z; the preceding bound remains valid, though it is not sharp.

Let $A = \{|Z| \leq u_q\}$. Decomposing over A and A^c,

$$\mathrm{FDR} = \mathbb{E}[\mathrm{FDP}\, \mathbf{1}_A] + \mathbb{E}[\mathrm{FDP}\, \mathbf{1}_{A^c}].$$

For the first term, integrate (35) over $z \in [-u_q, u_q]$ and use (38). For the second term, no conditional tail bound is needed: since $0 \leq \mathrm{FDP} \leq 1$,

$$\mathbb{E}[\mathrm{FDP}\, \mathbf{1}_{A^c}] \leq \mathbb{P}(A^c) = \mathbb{P}(|Z| > u_q) = 2\overline{\Phi}(u_q) = q.$$

Combining these two bounds yields

$$\mathrm{FDR} \leq q + \frac{q}{m} \sum_{i \in \mathcal{H}_0} \left(\frac{1 - q}{2} + \frac{u_q}{\sqrt{2\pi}}\right) = q \left\{1 + \pi_0 \left(\frac{1 - q}{2} + \frac{u_q}{\sqrt{2\pi}}\right)\right\},$$

as claimed. □

4 Discussion

The results above show that two-sided Gaussian testing under unrestricted correlation can have worse guarantees than the nominal BH level suggests. The BH procedure can have FDR larger than its nominal level by a factor of order $\sqrt{\log(1/q)}$ as $q \downarrow 0$, so no universal constant multiple of q can bound its FDR over all Gaussian correlation matrices. Because the lower-bound construction belongs to the common-factor class, Theorem 2 shows that the worst-case FDR within this class is of exact order $q\sqrt{\log(1/q)}$ as $q \downarrow 0$.

There are, however, other procedures with valid FDR guarantees for dependent two-sided Gaussian tests. One approach is the conditional-calibration framework of Fithian and Lei [2022]. The method decomposes FDR into hypothesis-wise contributions and, for each null i, conditions on a statistic that removes nuisance dependence while leaving a calibrated one-dimensional null distribution for the ith p-value. Their concrete dependence-adjusted BH procedure (dBH) calibrates a hypothesis-specific cutoff for the BH q-value and rejects H_i when its q-value falls below that calibrated cutoff. In general, dBH is at least as powerful as the Benjamini–Yekutieli procedure; under PRDS, it is at least as powerful as BH [Fithian and Lei, 2022]. Thus it provides finite-sample FDR control under known dependence without paying the full harmonic-factor correction of Benjamini and Yekutieli [2001].

Another route is the fixed-X knockoff filter of Barber and Candès [2015]. In a Gaussian linear model, knockoffs augment the design with synthetic variables that mimic the correlation structure of the original variables, and the resulting original-versus-knockoff comparisons create null statistics with the symmetry needed for FDR control. The correlated-Gaussian-testing interpretation is made explicit by Li and Fithian [2021], who recast fixed-X knockoffs as adding user-generated Gaussian noise to a Gaussian estimator to obtain a whitened estimator, followed by conditional inference. Knockoffs can use structural information and flexible feature-importance statistics unavailable to marginal testing. Their weakness is that whitening can require enough added noise to erase much of the signal when the original test statistics are strongly correlated, causing substantial power loss [Li and Fithian, 2021]. Knockoffs also face a threshold phenomenon when the rejection set is small. The calibrated knockoff method of Luo et al. [2025] uses conditional calibration to uniformly improve knockoff procedures, with especially large gains in small-rejection regimes.

More broadly, understanding FDR guarantees under dependence is essential because many real-world test statistics are well approximated by a multivariate Gaussian law. In such settings, the correlation matrix is often structured, but the induced two-sided p-values need not satisfy PRDS or the sign structure required by standard positive-dependence arguments. Two-sided testing can destroy monotonicity even when the underlying Gaussian variables have positive correlations. Procedures that explicitly use dependence information, and theory that clarifies when BH itself is reliable, are therefore both needed.

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